Exercises

Problems ($*$) are optionals.

Shell Model

1. ($*$) demonstrate Eq. (5) and (6).

2. ($*$) demonstrate the Schwartz inequality $|\langle a|b\rangle|^2 \leq ||a||^2||b||^2$.

3. ($*$) finish problem in Sect. 1.2.2, solving the even cases. Then consider the density current

$$\mathbf{j}(\mathbf{r}) = \frac{\hbar}{2im}[\psi(\mathbf{r})\boldsymbol{\nabla}\psi^*(\mathbf{r}) + \psi^*(\mathbf{r})\boldsymbol{\nabla}\psi(\mathbf{r})], \tag{1}$$

   and calculate how the current density behaves inside and outside the potential well.

4. Prove that the square of a general one–body operator is equal to a sum of one– and two–body operators.

5. Calculate the matrix elements of a two body operator Eq.(75) between two body states using Wick theorem.

6. In nuclear physics is sometimes assumed that is possible to approximate the interaction between one nucleon and all other nucleons by a potential $V(\mathbf{r})$. The Schrödinger equation with this potential is then solved, obtaining the eigenstates $\psi_i(\mathbf{r})$ and the corresponding energies ϵ_i. An antisymmetric A–particle wavefunction can be constructed using a Slated determinant,

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_1(\mathbf{r}_1) & \phi_2(\mathbf{r}_1) & \cdots & \phi_A(\mathbf{r}_1) \\ \phi_1(\mathbf{r}_2) & \phi_2(\mathbf{r}_2) & \cdots & \phi_A(\mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{r}_A) & \phi_2(\mathbf{r}_A) & \cdots & \phi_A(\mathbf{r}_A) \end{vmatrix}. \tag{2}$$

   Verify in the case of $A = 3$ that the Slater determinant is antisymmetric and that the formulation above is normalized. Show also that this implies that two (or more) particles cannot be found at the same place in this space. Note that this properties of a Slater determinant are valid also in the general $A \times A$ case and follow from the general properties of the determinant (as defined by the Leibnitz formula as the sum of possible permutations of the matrix elements with the appropriate phase).

7. Predict the level scheme of the first excitations of ^{50}Ti and ^{14}N and compare them to experiment on `https://www.nndc.bnl.gov/`.

8. (challenge) Derive Eq. (110) of the notes from Wigner-Eckart theorem and the rules of angular momentum coupling.

Collective modes

1. What is the parity of operator **s**, and why?

2. Derive single–particle (a.k.a. Weisskopf) units for electromagnetic transitions, cf. section 2.3 of Rowe.

3. Using Wick's theorem, calculate $\langle \alpha\beta^{-1}|H_0|\alpha'\beta'^{-1}\rangle$, with α particle state, β^{-1} a hole state, and H_0 a one–body operator (e.g. the one body part of an Hamiltonian).

4. Calculate the matrix element $Q_{\alpha\beta,2} = \langle \alpha\beta^{-1}|\hat{Q}_2|0\rangle$ considering $\alpha = d_{5/2}$ state in an Harmonic oscillator, and β the $s_{1/2}$ state, and $\hat{Q}_2 = e\sum_\mu r^2 Y^2_\mu(\hat{\theta})$ the quadrupole electric operator. Feel free to use the phenomenological estimate $\langle r^\lambda\rangle = 3R_0^\lambda/(\lambda+3)$, with R_0 the average radius, where you can use the relation $R_0 = r_0 A^{1/3}$ with $r_0 = 1.2$ fm.

   Hint: The reduced matrix element of a multipole operator is given by

$$\begin{aligned}\langle a||Q_\lambda||b\rangle =& \frac{eR^\lambda_{ab}}{\sqrt{2\pi}}(-1)^{1/2+j_b-\lambda}\frac{1+(-1)^{l_a+l_b+\lambda}}{2} \\ & \sqrt{2\lambda+1}\sqrt{2j_a+1}\sqrt{2j_b+1}\begin{pmatrix} j_a & j_b & \lambda \\ 1/2 & -1/2 & 0 \end{pmatrix}\end{aligned} \tag{3}$$

   with R^λ_{ab} the radial part of the matrix element. If you wish, can derive the multipole excitation operator using Wigner-Eckart theorem.

5. Using the matrix element calculated in the previous exercise,

$$\frac{1}{\chi} = \sum_{\alpha\beta}\frac{|Q_{\alpha\beta,2}|^2}{\epsilon_\alpha - \epsilon_\beta - \hbar\omega}, \tag{4}$$

   calculate the energy of the resulting phonon considering $1/\chi = 1$ MeV, $\epsilon_\alpha - \epsilon_\beta = 2$ MeV

6. $(*)$ Derive Tamm–Dancoff approximation using the equation of motion method.

7. Recover Tamm–Dancoff approximation by imposing $Y = 0$ in the RPA matrix equation.

8. Use the matrix element calculated in exercise number 4 and the valence space of $d_{5/2}$ and $s_{1/2}$ levels to diagonalize the RPA equation and find the components X and Y of the phonon, and compare the result with the one obtained by Tamm–Dancoff approximation.