Hilbert Spaces

1 Introduction

Hilbert spaces are the mathematical structures that underpinns modern quantum mechanics. Here we go through the essential concepts needed before learning how to apply this to the study of nature. Familarity with mathematical symbols, calculus and linear algebra is assumed. Some of the notation we will use are:

$\Rightarrow$	leads to
$\forall$	for all
$\in$	belongs to
$\longrightarrow$	maps
$\subseteq$	subset of
$\Leftrightarrow$	equivalent
$\rightarrow$	goes to
$\mid$	with the condition

2 Linear spaces

We want to consider the topic of **linear spaces**. In linear algebra, we have the following composition rules for vectors. Note that the latin letters $u, v, w, ...$ denote vectors while the greek letters $\lambda, \mu, ...$ denote scalars.

(L1)	$u + v = v + u$	(commutative law)
	$u + (v + w) = (u + v) + w$	(associative law)
	$u + 0 = u$	(0 is the null element)
	$u + (-1)u = 0$	(-u is the opposite element)
(L2)	$\lambda(\mu u) = (\lambda\mu)u$	
	$1 \cdot u = u$	
	$0 \cdot u = 0$	
	$\lambda \cdot 0 = 0$	(0 is the null element)
(L3)	$(\lambda + \mu)u = \lambda u + \mu u$	(distributive law)
	$\lambda(u + v) = \lambda u + \lambda v$	(distributive law)

Remark: The same rules apply for other entities, such as n-tiples in $\mathbb{R}^n$ and $m \times n$-matrices. For n-tiples, addition is defined as

$$(x_1, x_2, \ ... \ , x_n) + (y_1, y_2, \ ... \ , y_n) = (x_1 + y_1, x_2 + y_2, \ ... \ , x_n + y_n). \quad (1)$$

and multiplication with a scalar is defined as

$$\lambda(x_1, x_2, \ ... \ , x_n) = (\lambda x_1, \lambda x_2, \ ... \ , \lambda x_n) \quad (2)$$

The rules (L1)-(L3) also apply to functions, given appropriate definitions of the operations 'addition' and 'multiplication with scalar'. Such spaces are called **function spaces**.

We now define the term linear space, a space in which (L1)-(L3) are axioms and further properties are derived from these axioms.

DEFINITION: A **linear space over** $\mathbb{R}$ is a set $\mathcal{H}$ on which the following operations are defined:

$$\begin{array}{ll} \textit{addition:} & u \in \mathcal{H},\, v \in \mathcal{H} \Longrightarrow u + v \in \mathcal{H}, \\ \textit{multiplication with scalar:} & \lambda \in \mathbb{R},\, u \in \mathcal{H} \Longrightarrow \lambda u \in \mathcal{H}, \end{array}$$

such that (L1)-(L3) applies. The space can also be over $\mathbb{C}$ instead of $\mathbb{R}$ (we then talk about a **linear space over** $\mathbb{C}$). The elements on $\mathcal{H}$ are called **vectors**. Instead of linear space, the term **vector space** may be used.

EXAMPLE: For the complex n-tiples

$$\mathbb{C}^n = \{(z_1, z_2, \dots, z_n) | z_k \in \mathbb{C},\ (k = 1, \dots, n)\}$$

addition and multiplication with a scalar are defined as in (1) and (2). It can be shown that (L1)-(L3) applies to this set, so $\mathbb{C}^n$ is a linear space over $\mathbb{C}$.

EXAMPLE: A subset Ω of $\mathbb{R}$ can be a limited or unlimited interval of $\mathbb{R}$. Given end points a and b, some possible subsets Ω are:

$$(a, b) = \{x | a < x < b\}\ ,\ (a, b] = \{x | a < x \leq b\}\ ,\ [a, \infty) = \{x | a \leq x\}$$

Let

$$\Phi = \Phi(\Omega) = \{\text{all functions } f : \Omega \longrightarrow \mathbb{R}\}$$

[1]be the set of all real functions on Ω. For two functions $f, g \in \Phi(\Omega), \lambda \in \mathbb{R}$, the functions $f + g$ and λf are defined as

$$f + g : x \mapsto f(x) + g(x),\ x \in \Omega,$$

$$\lambda f : x \mapsto \lambda f(x),\ x \in \Omega,$$

Using these operations, it can be shown that (L1)-(L3) applies to Φ and it is therefore a linear space over $\mathbb{R}$. (Here, 0 represents the null function.) This is an example of a function space. We simply regard the functions as vectors in the linear space $\Phi(\Omega)$.

If one replaces $\mathbb{R}$ with $\mathbb{C}$, it can be shown that the set of all complex functions on Ω is a linear space over $\mathbb{C}$.

DEFINITION: $\mathcal{U} \subseteq \mathcal{H}$ is a **linear subspace** of $\mathcal{H}$ if

$$u \in \mathcal{U}, v \in \mathcal{U} \Longrightarrow u + v \in \mathcal{U},$$

and

$$\lambda \in \mathbb{R} \text{ (or } \mathbb{C}), u \in \mathcal{U} \Longrightarrow \lambda u \in \mathcal{U}.$$

Because $\mathcal{U}$ is a subspace of $\mathcal{H}$, (L1)-(L3) still apply.

EXAMPLE: If $\mathcal{H}$ is a set containing all geometrical vectors in 3-dimensional space and $\mathcal{U}$ is a set containing all vectors parallel to a given plane π, then $\mathcal{U}$ is a subspace of $\mathcal{H}$. It can be shown that if u and v are two vectors in the plane π, both the vector $u + v$ and all vectors λu, with $\lambda \in \mathbb{R}$ are also in π.

[1]The notation should be read as: The set of all functions f that maps points from the interval Ω to $\mathbb{R}$

EXAMPLE: The set

$$\mathcal{U} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{k=1}^{n} z_k = 0\}$$

is a subspace of $\mathbb{C}^n$, because

$$z' \in \mathcal{U}, z'' \in \mathcal{U} \Longrightarrow \sum_{k=1}^{n}(z'_k + z''_k) = \sum_{k=1}^{n} z'^k + \sum_{k=1}^{n} z''_k = 0\},$$

$$\lambda \in \mathbb{C}, z \in \mathcal{U} \Longrightarrow \sum_{k=1}^{n} \lambda z_k = 0$$

However, the set

$$\mathcal{U} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{k=1}^{n} z_k = 1\}$$

is *not* a subspace of $\mathbb{C}^n$. This is because the zero vector $0 = (0, ..., 0) \notin \mathcal{U}$. All subspaces must contain the zero vector, because $0 \cdot z = 0$ for all z.

EXAMPLE: The set

$$\mathcal{U} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$$

is *not* a subspace of $\mathbb{R}^2$. We can easily prove this with an example:

$$x' = (0, 1) \in \mathcal{U},\ x'' = (1, 0) \in \mathcal{U}$$
$$\Longrightarrow x' + x'' = (1, 1) \notin \mathcal{U}$$

EXAMPLE: The set $\Pi_n(\mathbb{R})$ containing all possible polynomials

$$\sum_{k=0}^{n} a_k x^k$$

with a degree $\leq n$ and real coefficients a_k can be shown to be a subspace of $\Phi(\mathbb{R})$. If p and q are polynomials and $\lambda \in \mathbb{R}$, both $p + q$ and λp are polynomials as well. This also holds true for polynomials with complex coefficients, only in this case $\Pi_n(\mathbb{C})$ is a subspace of $\Phi(\mathbb{C})$.

DEFINITION: Consider a linear space $\mathcal{H}$. The vectors $u_1, u_2, ..., u_n \in \mathcal{H}$ are said to be **linearly independent** if

$$\sum_{j=1}^{n} \lambda_j u_j = 0 \Longrightarrow \lambda_j = 0,\ \forall j$$

If any vector $v \in \mathcal{H}$ can be expressed as a linear combination of the vectors u_j, the set $u_1, \dots, u_n$ is said to be a **basis** in $\mathcal{H}$.

Remark: This is certainly true in a finite space. To show this for an infinite space, more care is required.

All bases in a linear space $\mathcal{H}$ have the same number of elements. The number of elements in a basis of $\mathcal{H}$ is the **dimension** of $\mathcal{H}$. It is not possible to have a finite basis if the space is infinite-dimensional.

EXAMPLE: The dimension of n-dimensional real space is

$$\dim \mathbb{R}^n = n.$$

The dimension of n-dimensional complex space is

$$\dim \mathbb{C}^n = n$$

The dimension of the set containing all n-degree polynomials is

$$\dim \Pi_n = n + 1,$$

because the basis of Π_n is $[1, x, x^2, ..., x^n]$. The set Π_n is a subspace of all polynomials (any degree), which is denoted Π.

We will now define the important function spaces $\mathcal{C}(\Omega)$ and $\mathcal{C}^k(\Omega)$. Ω is a connected domain in $\mathbb{R}^n$, that may be open (the boundary points are not part of the domain) or closed (all boundary points are part of the domain).

- $\mathcal{C}(\Omega)$: Functions that are continuous in Ω. This is a linear space and a subspace of $\Phi(\Omega)$.
- $\mathcal{C}^k(\Omega)$: Functions whose derivatives of order $\leq k$ are continuous in Ω. This is a linear space and a subspace of $\Phi(\Omega)$.

3 Scalar product and norm

Given two elements in a linear space u, v we denote their **scalar product** by $(u|v)$. In the ordinary $\mathbb{R}^n$ case, the scalar product can be written

$$(u|v) = \sum_{k=1}^{n} u_k v_k. \tag{3}$$

The length of a vector can be expressed with the help of a scalar product:

$$\|u\| = (u|u)^{1/2} = (\sum_{k=1}^{n} u_k^2)^{1/2} \tag{4}$$

The length of a vector is usually called the **norm**. The norm $\|u\| \geq 0$ for all $u \in \mathcal{H}$.
In the $\mathbb{C}$ case, we must modify the definition of the scalar product. The norm should still be a real, positive number and be defined by the scalar product. Equation (4) will usually *not* give a real number if the numbers u_k are complex. If the scalar product is defined as

$$(u|v) = \sum_{k=1}^{n} u_k^* v_k, \tag{5}$$

then the norm can be written

$$\|u\| = (u|u)^{1/2} = (\sum_{k=1}^{n} |u_k|^2)^{1/2}.$$

This is a real number ≥ 0. The following rules hold true for the scalar product $(u|v)$, as defined by (3) and (5):

(S1) $(u|\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 (u|v_1) + \lambda_2 (u|v_2)$

(S2) $(u|v) = (v|u)^*$

(S3) $(u|u) \geq 0$ (equality for $u = 0$)

(S4) $(\lambda_1 u_1 + \lambda_2 u_2|v) = \lambda_1^* (u_1|v) + \lambda_2^* (u_2|v))$

Note that (S4) follows from (S1) and (S2).

DEFINITION: A **scalar product** on a linear space $\mathcal{H}$ is a rule which associates two elements $u, v \in \mathcal{H}$ to a scalar, $(u|v)$, so that the rules (S1)-(S4) apply. A linear space with a scalar product is called a **pre-Hilbert space**.

EXAMPLE: In the continuous case $\mathcal{C}(\Omega)$ the scalar product is defined by

$$(u|v) = \int_\Omega u^*(x)v(x)dx.$$

It is easily verified that rules (S1)-(S4) apply.

EXAMPLE: Now for the general case. Given a positive function $w > 0, w \in \mathcal{C}(\Omega)$,

$$(u|v)_w = \int_\Omega u^*(x)v(x)w(x)dx \tag{6}$$

is a scalar product on $\mathcal{C}(\Omega)$. When Ω is a finite domain in $\mathbb{R}$ this also holds for piecewise continuous functions with finite-value discontinuities. The function w is called a **weight function**. Different weight functions define different pre-Hilbert spaces.

DEFINITION: For a pre-Hilbert space $\mathcal{H}$,

i) u, v are **orthogonal** if $(u|v) = 0$. This is written $u \perp v$.

ii) the **norm** of u is defined as $\|u\| = (u|u)^{1/2}$.

For a scalar product with a weight function according to (6), orthogonality is given by

$$u \perp v \Longrightarrow \int_\Omega u^*(x)v(x)w(x)dx = 0$$

and the norm of a vector u is given by

$$\|u\| = (\int_\Omega |u(x)|^2 w(x)dx)^{1/2}$$

We will now look at a few important properties of norms. *Pythagoras' theorem*

$$u \perp v \Longrightarrow \|u + v\|^2 = \|u\|^2 + \|v\|^2$$

is valid in all pre-Hilbert spaces. In linear algebra we have the expression

$$(u|v) = \|u\| \cdot \|v\| \cos\theta \Longrightarrow |(u|v)| \leq \|u\| \cdot \|v\|,$$

where equality means u and v are parallel. This expression is known as *Cauchy's inequality* or *Cauchy-Schwarz' inequality*. For the special case of the scalar product defined by (5), Cauchy's inequality reads

$$|\sum_{k=1}^{n} u_k^* v_k| \leq (\sum_{k=1}^{n} (u_k)^2)^{1/2} (\sum_{k=1}^{n} (v_k)^2)^{1/2}$$

For the special case of the scalar product defined by (6), Cauchy's inequality reads

$$|\int_\Omega u^*(x)v(x)w(x)dx| \leq (\int_\Omega |u(x)|^2 w(x)dx)^{1/2} (\int_\Omega |v(x)|^2 w(x)dx)^{1/2}$$

For $\|u\| = (u|u)^{1/2}$ the following rules hold:

(i) $\|u\| \geq 0$ ($\|u\| = 0$ only if $u = 0$)

(ii) $\|\lambda u\| = |\lambda| \cdot \|u\|$

(iii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

The space of continuous functions $\mathcal{C}(\Omega)$ with scalar product

$$(u|v)_w = \int_\Omega u^*(x)v(x)w(x)dx \tag{7}$$

and norm

$$\|u\| = (\int_\Omega |u(x)|^2 w(x)dx)^{1/2} \tag{8}$$

is a pre-Hilbert space. In general, for a function space to be a pre-Hilbert space the functions must not necessarily be continuous but the integrals in (7) and (8) must exist and be finite.

We will now introduce the function space L_2. Let $w(x) > 0 \in \Omega$. With $L_2 = L_2(w, \Omega)$ we refer to the set of functions in Ω such that

$$\|u\|_{L_2(w)} = (\int_\Omega |u(x)|^2 w(x)dx)^{1/2}$$

exist and is finite. The special case of $w = 1 \Longrightarrow L_2 \equiv L_2(\Omega)$. The *existence* and *finiteness* of the integral is a rather subtle question. This is left unexplored by us and we will assume, unless explicitly stated, that this is always the case with the functions we consider.

It is not immediately obvious that $L_2(w, \Omega)$ is a pre-Hilbert space. One can show that $L_2(w, \Omega)$ have all the properties required to be a linear space (see Sparr, page 257). However, complications arise when trying to show that it is also a pre-Hilbert space. Consider the generalized scalar product in (6). There are some functions such that the scalar product

$$(u|u)_w = \int_\Omega |u|^2 w(x)dx = 0,$$

even though u is not the zero function. This does not comply with rule (S3). One such function is

$$u(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

The work-around for this problem is to *identify* these functions with the zero function. With this extension of the definition of zero function, the $L_2(w, \Omega)$ becomes a pre-Hilbert space.

EXAMPLE:

$1 \in L_2([0,1])$	$1 \notin L_2(\mathbb{R})$
$x^{-1/3} \in L_2([0,1])$	$x^{-1/3} \notin L_2([1,\infty))$
$e^{-x} \in L_2([0,\infty))$	$e^{-x} \notin L_2(\mathbb{R})$

Check that these are true by considering ordinary integrals and see if they diverge. Notice that here $L_2(\Omega)$ means $L_2(1,\Omega)$.

4 Projections

We define a **projection of u on v** as

$$P_{[v]}u = \frac{(v|u)}{\|v\|^2}v.$$

In the case of u and v being geometrical vectors, the projection of u on v is the component of u along the direction of v. The concept of projection is now exemplified in the context of Fourier series, together with the notion of orthogonality and norm.

EXAMPLE: Consider $L_2([-\pi,\pi])$.

$$(e^{ikx}|e^{inx}) = \int_{-\pi}^{\pi} e^{-ikx}e^{inx}dx = \int_{-\pi}^{\pi} e^{i(n-k)x}dx =$$
$$= \frac{1}{i(n-k)}\left[e^{i(n-k)x}\right]_{-\pi}^{\pi} = 0 \text{ if } n \neq k$$
$$\Longrightarrow (e^{ikx}|e^{inx}) = \begin{cases} 2\pi & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

In other words, e^{ikx} and e^{inx} are orthogonal functions if $n \neq k$. Now, let u be a given function in $L_2([-\pi,\pi])$. We find that

$$(e^{ikx}|u) = \int_{-\pi}^{\pi} e^{-ikx}u(x)dx = 2\pi c_k(u),$$

where $c_k(u)$ is a Fourier coefficient for u. The projection of u on the subspace spanned by the functions e^{ikx} is

$$P_{[e^{ikx}]}u = \frac{(e^{ikx}|u)}{\|e^{ikx}\|^2}e^{ikx} = c_k(u)e^{ikx}.$$

This is a term in the Fourier series of u. The complete Fourier series is written

$$u \sim \sum_{-\infty}^{\infty} c_k(u)e^{ikx} = \sum_{-\infty}^{\infty} \frac{(e^{ikx}|u)}{\|e^{ikx}\|^2}e^{ikx}.$$

(Here the symbol $\sim$ is used instead of the equal sign $=$ since we have not yet discussed the criteria for convergence, an issue that we not consider here. Essentially the equal sign can be used if we limit ourselves to continous functions with continuous derivatives, see Sparr). Evidently, the terms of the Fourier series of u can be interpreted as u:s projections on the orthogonal functions e^{ikx}. We have found a geometrical interpretation of Fourier series.

EXAMPLE: The previous example can be used to show that the functions

$$\{\cos(kx)\}_{k=0}^{\infty} \cup \{\sin(kx)\}_{k=1}^{\infty}$$

are pairwise orthogonal (verify this by expanding $\cos(kx)$ and $\sin(kx)$ in the exponential functions e^{ikx} and e^{-ikx}). L_2-norms of the following functions are

$$\|1(x)\| = \left(\int_{-\pi}^{\pi} |1|^2 dx\right)^{1/2} = \sqrt{2\pi}$$

$$\|\cos(kx)\| = \left(\int_{-\pi}^{\pi} \cos^2(kx) dx\right)^{1/2} = \sqrt{\pi},\ k = 1, 2, ...$$

$$\|\sin(kx)\| = \left(\int_{-\pi}^{\pi} \sin^2(kx) dx\right)^{1/2} = \sqrt{\pi},\ k = 1, 2, ...$$

The norm of $1(x)$ differs from the other norms. For an arbitrary function u, it holds that

$$(\cos(kx)|u) = \int_{-\pi}^{\pi} \cos(kx)u(x)dx = \pi a_k(u),\ k = 0, 1, ...$$

$$(\sin(kx)|u) = \int_{-\pi}^{\pi} \sin(kx)u(x)dx = \pi b_k(u),\ k = 1, 2, ...$$

where $a_k(u)$ and $b_k(u)$ are the trigonometric Fourier coefficients of u. The projections of u on the subspaces $[\cos(kx)]$ and $[\sin(kx)]$ are

$$P_{[1(x)]}u = P_{[\cos(0x)]}u = \frac{(1(x)|u)}{\|1(x)\|^2}1(x) = \frac{a_0(u)}{2}$$

$$P_{[\cos(kx)]}u = \frac{(\cos(kx)|u)}{\|\cos(kx)\|^2}\cos(kx) = a_k(u)\cos(kx),\ k = 1, 2, ...$$

$$P_{[\sin(kx)]}u = \frac{(\sin(kx)|u)}{\|\sin(kx)\|^2}\sin(kx) = b_k(u)\sin(kx),\ k = 1, 2, ...$$

We recognize these projections as the terms in the trigonometric Fourier series. We can now arrange things according to Fourier theory:

$$u \sim \frac{a_0(u)}{2} + \sum_{k=1}^{\infty}(a_k(u)\cos(kx) + b_k(u)\sin(kx)) =$$

$$= \sum_{k=0}^{\infty} P_{[\cos(kx)]}u + \sum_{k=1}^{\infty} P_{[\sin(kx)]}u$$

and we note that, from linear algebra, if $\varphi_1, ..., \varphi_n$ is an orthogonal basis in $\mathbb{R}^n$, every $u \in \mathbb{R}^n$ can be written

$$u = \sum_{k=1}^{n} \frac{(\varphi_k|u)}{\|\varphi_k\|^2}\varphi_k.$$

5 Gram-Schmidt's orthogonalization method

It is advantageous to use orthogonal bases in pre-Hilbert spaces. *Gram-Schmidt's orthogonalization method* is a method for constructing an orthogonal basis from a non-orthogonal basis.

THEOREM: From a set of linearly independent vectors $u_1, ..., u_n$, one can always construct a set of orthogonal vectors $\varphi_1, ..., \varphi_n$ that are linear combinations of

$u_1, ..., u_n$. These vectors are unambiguously defined except for a proportionality factor (i.e. an arbitrary multiplicative scalar).

The general proof of the theorem is based on the induction method, and one finds that

$$\varphi_n(x) = u_n(x) - \sum_{m=1}^{n-1} \frac{(\varphi_m|u_n)}{\rho_m}\varphi_m(x), \qquad \rho_m = \|\varphi_m\|^2.$$

Here, we just consider a concrete example to show how to proceed in practice. Consider $L_2([-1,1])$ and the polynomials $u_k(x) = x^k, k = 0, 1, ...$ (these are called *monomials*). The orthogonal functions φ_k are calculated in the following manner:

$$\varphi_0 = u_0 = 1,$$

$$\rho_0 = (\varphi_0|\varphi_0) = \int_{-1}^{1} dx = 2,$$

$$\varphi_1 = u_1 - P_{[\varphi_0]}u_1 = u_1 - \frac{1}{\rho_0}(\varphi_0|u_1)\varphi_0 = x - \frac{1}{2}(\int_{-1}^{1} xdx)1 = x,$$

$$\rho_1 = (\varphi_1|\varphi_1) = \int_{-1}^{1} x^2 dx = \frac{2}{3},$$

$$\varphi_2 = u_2 - P_{[\varphi_0,\varphi_1]}u_2 = u_2 - \frac{1}{\rho_0}(\varphi_0|u_2)\varphi_0 - \frac{1}{\rho_1}(\varphi_1|u_2)\varphi_1 =$$

$$= x^2 - \frac{1}{2}(\int_{-1}^{1} x^2 dx)1 - \frac{3}{2}(\int_{-1}^{1} x^3 dx)x = x^2 - \frac{1}{3},$$

$$\rho_2 = (\varphi_2|\varphi_2) = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \frac{8}{45},$$

etc.

In the previous example, we orthogonalized monomials according to the scalar product

$$(u|v) = \int_{\Omega} u^*(x)v(x)dx.$$

It possible to obtain different orthogonal polynomials by using Gram-Schmidt's method to orthogonalize monomials for different pre-Hilbert spaces $L_2(w, I)$. There is a group of very important orthogonal polynomials which recur naturally in different applications. They are available as standard functions in programs such as Fortran, Mathematica, Matlab and Maple. We will only mention a few of these especially important orthogonal polynomials here.

Legendre polynomials: Let $I = [-1, 1]$ and $w(x) = 1$. These are the polynomials that appeared in the example above. They are usually "normalized" by the condition $P(1) = 1$. The first five Legendre polynomials are:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$

Chebyshev polynomials: Let $I = [-1, 1]$ and $w(x) = 1/\sqrt{1 - x^2}$. The obtained polynomials can be described by the formula

$$T_n(x) = \cos(n \cdot \arccos(x)),$$

which can be shown to be a polynomial using trigonometric formulas. The first five Chebyshev polynomials are

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

Chebyshev polynomials can used to describe the quantum time evolution of a wavefunction according to the time-dependent Schrödinger equation.

Laguerre polynomials: Let $I = [0, \infty)$ and $w(x) = e^{-x}$.The obtained polynomials can be described by the formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n).$$

The first four Laguerre polynomials are

$$L_0(x) = 1$$

$$L_1(x) = -x + 1$$

$$L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$$

$$L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$$

The Laguerre polynomials, which satisfy the differential equation

$$xy" + (\alpha + 1 - x)y' + ny = 0$$

with $\alpha = 0$, appear in the radial part of the solution to the Schrödinger equation for a 1-electron atom. The $\alpha \neq 0$ case corresponds to the so-called generalized Laguerre polynomials.

Hermite polynomials: Let $I = \mathbb{R}$ and $w(x) = e^{-x^2}$. The obtained polynomials can be described by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

The first four Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

The series of Hermite polynomials can also be calculated by recursion:

$$H_{n+1} = 2xH_n(x) - 2nH_{n-1}(x)$$

Hermite polynomials have the symmetry condition:

$$H_n(-x) = (-1)^n H_n(x)$$

The normalization is

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \delta_{n,m}2^n n!\sqrt{\pi}.$$

One can show that a modified function

$$\Psi_n(x) = (2^n n!\sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x)$$

satisfies

$$\int_{-\infty}^{\infty} \Psi_n(x)\Psi_m(x)dx = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \delta_{m,n}.$$

These functions are important because they provide the solution to

$$\Psi_n''(x) + (2n + 1 - x^2)\Psi_n(x) = 0.$$

This is the Schrödinger equation for the quantum harmonic oscillator.

6 Convergence in norm

A norm in a linear space permits a natural convergence concept:

$$u_n \to u \text{ when } n \to \infty \iff \|u_n - u\| \to 0 \text{ when } n \to \infty \tag{9}$$

Let us study convergence in linear spaces. Consider the set $\{\varphi_k\}_1^\infty$ of pairwise orthogonal vectors in $\mathcal{H}$. We wish to determine if we can write, for a generic $u \in \mathcal{H}$,

$$u = \sum_{k=1}^{\infty} c_k \varphi_k,$$

and if yes, how to choose the coefficients c_k. The convergence criteria in (9) states

$$\|u - \sum_{k=1}^{N} c_k \varphi_k\| \to 0 \text{ when } N \to \infty$$

It can be shown that the optimal choice of coefficient is

$$c_k = \frac{1}{\rho_k}(\varphi_k|u),\ \rho_k = (\varphi_k|\varphi_k).$$

DEFINITION: Suppose $\{\varphi_k\}_1^\infty$ is a sequence of pairwise orthogonal vectors in $\mathcal{H}$. The quantities

$$c_k(u) = \frac{1}{\rho_k}(\varphi_k|u),\ \rho_k = (\varphi_k|\varphi_k)$$

are called the Fourier coefficients with respect to $\{\varphi_k\}_1^\infty$. The series

$$u \sim \sum_{k=1}^{\infty} c_k(u)\varphi_k \tag{10}$$

is called the Fourier series of u, or the orthogonal expansion with respect to $\{\varphi_k\}_1^\infty$.

Orthogonality does *not* guarantee that the series (10) converges for all u. It is also required that $\{\varphi_k\}_1^\infty$ contains "enough" function to span all of $\mathcal{H}$, as presented in the next definition:

DEFINITION: $\{\varphi_k\}_1^\infty$ is an **orthogonal basis** in $\mathcal{H}$ if the vectors are pairwise orthogonal and if any $u \in \mathcal{H}$ can be developed in a Fourier series:

$$u = \sum_{1}^{\infty} \frac{1}{\rho_k}(\varphi_k|u)\varphi_k,\ \rho_k = (\varphi_k|\varphi_k)$$

One also says that $\{\varphi_k\}_1^\infty$ is a **complete orthogonal system**. The basis is also **orthonormal** when $\rho_k = 1, \forall k$.

For our scopes, the spaces $L_2(w, I)$ have bases. The systems of polynomials and trigonometric functions previously mentioned make up bases in L_2.

A **Hilbert space** is a pre-Hilbert space which is also complete. The space L_2 is a pre-Hilbert space that can also be shown to be complete (Fischer-Reisz's theorem). Therefore, $L_2(w, I)$ is a Hilbert space.

Completeness is not a trivial property; consider for example the space of continuous function $\mathcal{C}^0$ and a sequence within it defined as

$$f_k(x) = \begin{cases} 1 & \text{if } \frac{1}{k} < x \leq 1 \\ \frac{kx+1}{2} & \text{if } -\frac{1}{k} \leq x \leq \frac{1}{k} \\ 0 & \text{if } -1 \leq x < -\frac{1}{k}, \end{cases}$$

with $k = 2, 3, .., \infty$. If, for simplicity, we consider the weight function $w(x) = 1$, we see that two functions $f_k(x), f_l(x)$ are such that

$$\lim_{k,l\to\infty}\int_{-1}^{1}|f_k(x)-f_l(x)|^2dx=0.$$

This means that the distance between $f_k, f_l \to 0$ as $x \to \infty$. Thus there is a limit to the sequence, i.e. $f_{k\to\infty}$ tends to some function. However, such limit does not belong to $\mathcal{C}^0$, since

$$f_{k\to\infty}(x)=f(x)\begin{cases}0 & \text{if } -1\le x<-0\\ 1 & \text{if } 0<x\le 1\end{cases}$$

is not a continuous function. Thus $\mathcal{C}^0$ is not "big" enough to be complete (this issue does not exist for finite spaces; once given a scalar product, they are Hilbert spaces). To have completeness, one must enlarge the space of admissible functions, e.g. with discontinuities. To handle this "enlarged" space, one needs to use a more general concept of integration (the so-called Lebesgue integration), which is not further discussed here.

Examples of bases in $L_2(w)$:

- The sequence of vectors $1, x, x^2, ...$ is a basis of $L_2[a,b]$ (this is supported by the *Weierstrass theorem*, see Sparr, page 280).
- On the interval $[-1,1]$, some orthogonal bases are the Legendre polynomials, $\{\sin(k\pi x)\}_1^\infty \cup \{\cos(k\pi x)\}_0^\infty$ and $\{e^{ikx}\}_{-\infty}^{\infty}$.
- More general, the Chebyshev, Laguerre and Hermite polynomials can be used as orthogonal bases in order to expand functions.

7 Operators in Hilbert space

An operator produces an image of a Hilbert space by mapping elements of one vector space to another vector space:

$$\mathcal{A}:\mathcal{H}_1\to\mathcal{H}_2$$

From linearity follows that

$$\mathcal{A}(\lambda u+\mu v)=\lambda\mathcal{A}u+\mu\mathcal{A}(v).$$

An important case is when $\mathcal{H}_1=\mathcal{H}_2=\mathcal{H}$, i.e. we have an operator on $\mathcal{H}$.

EXAMPLE: The $n\times n$ matrix

$$\mathsf{A}\in\mathbb{R}^{n\times n}$$

defines an operator $\mathbb{C}^n\to\mathbb{C}^n$ so that

$$\mathsf{y}=\mathsf{Ax}$$

EXAMPLE: (Integral operators.) Consider a compact interval I on $\mathbb{R}$ and an operator $K(x,y)$ that is continuous on $\mathbb{R}$.

$$v(x)=\int_I K(x,y)u(y)dy$$

This is reminiscent of the matrix product $v_i=\sum_j A_{ij}u_j$, only with an integral instead of a sum. The function $K(x,y)$ is called the *kernel* of the operator.

Let's consider a pre-Hilbert space $\mathcal{H}$. An operator $\mathcal{A}$ is said to be **bounded** if

$$\|\mathcal{A}u\| \leq c\|u\|, \ \forall u \in \mathcal{H}.$$

That is, for all u's, the norm is bounded by the same number $c \in \mathbb{R}$. Bounded operators are continuous in the sense that

$$x_n \to x \implies \mathcal{A}x_n \to \mathcal{A}x,$$

which follows from:

$$\|\mathcal{A}x_n - \mathcal{A}x\| = \|\mathcal{A}(x_n - x)\| \leq c\|x_n - x\|.$$

What about differential operators? Even in simple cases, we encounter problems with unbounded operators.

EXAMPLE: Consider the differential operator $D : u \mapsto u'$, an operator of the type

$$D : \mathcal{C}^1 \to \mathcal{C}^0.$$

Consider $u(x) = e^{ikx}$. We now have $Du(x) = ike^{ikx} = iku(x)$. Independent of which norm is used,

$$\|Du\| = k\|u\|$$

Since k can be any number, D is *not* bounded.

8 Symmetric operators

Consider a square matrix A. If

$$\mathsf{Au} = \lambda\mathsf{u}, \ \mathsf{u} \neq 0,$$

then λ is the *eigenvalue* and u is the *eigenvector* of A. If A has a basis of eigenvectors $\mathsf{e}_1, ..., \mathsf{e}_n$, then

$$\mathsf{u} = \sum_{k=1}^{n} u_k \mathsf{e}_k \Longrightarrow \mathsf{v} = \mathsf{Au} = \sum_{k=1}^{n} u_k \mathsf{Ae}_k = \sum_{k=1}^{n} \lambda_k u_k \mathsf{e}_k$$

explains the action on a generic u. This can be written on matrix form as

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

This shows that in a basis of eigenvectors, A is represented by a diagonal matrix. If A is real and symmetric,

$$\mathsf{A}^\mathsf{T} = \mathsf{A}, \ \mathsf{A} \in \mathbb{R}^{n \times n}$$

then according to the spectral theorem *all eigenvalues* λ_k *are real* and A has an *orthonormal basis of eigenvectors.*

In the orthonormal basis $\{\mathsf{e}_k\}_1^n$ of A,

$$u_k = (\mathsf{e}_k|\mathsf{u}), \ u_k\mathsf{e}_k = \frac{(\mathsf{e}_k|\mathsf{u})}{\|\mathsf{e}_k\|^2}\mathsf{e}_k = P_{[\mathsf{e}_k]}\mathsf{u},$$

$$\implies \mathsf{Au} = \sum_{k=1}^{n} \lambda_k P_{[\mathsf{e}_k]} \mathsf{u},$$

where the operator $P_{[\mathsf{e}_k]}$ is the orthogonal projection on the eigenvector e_k and $\lambda_k \in \mathbb{R}$ is the corresponding eigenvalue, $k = 1, 2, ...$ Since this is valid for any u, another way to express the spectral theorem is

$$\mathsf{A} \equiv \sum_{k=1}^{n} \lambda_k P_{[\mathsf{e}_k]}.$$

A real, symmetric matrix A is **positive definite** or **positive semi-definite** if it has the quadratic form

$$\mathsf{u}^\mathsf{T}\mathsf{Au} > 0 \text{ or } \mathsf{u}^\mathsf{T}\mathsf{Au} \geq 0, \ \forall \mathsf{u} \neq 0.$$

This is equivalent with

$$\lambda_k > 0 \text{ or } \lambda_k \geq 0, \ \forall k.$$

This can be generalized for *complex matrices* with a few changes. We define the operation $\widetilde{\mathsf{A}}$ as a simultaneous conjugation and transposition:

$$\widetilde{\mathsf{A}} = (\mathsf{A}^*)^\mathsf{T}$$

We want to show that $\widetilde{\mathsf{A}} = \mathsf{A}$ when

$$(\mathsf{Au}|\mathsf{v}) = (\mathsf{u}|\mathsf{Av}), \ \forall \mathsf{u}, \mathsf{v} \in \mathbb{R}^n \text{ or } \mathbb{C}^n \tag{11}$$

We show this by analysing the two sides of (11) separately.

$$\text{Left side} = (\mathsf{Au}|\mathsf{v}) = \sum_{k=1}^{n} (\mathsf{Au})_k^* v_k = \sum_{k=1}^{n} (\sum_{j=1}^{n} A_{kj} u_j)^* v_k = \sum_{j,k=1}^{n} u_j^* A_{kj}^* v_k$$

$$\text{Right side} = (\mathsf{u}|\mathsf{Av}) = \sum_{j=1}^{n} u_j^* (\mathsf{Av})_j = \sum_{j,k=1}^{n} u_j^* A_{jk} v_k$$

Now since both sides should be equal for any choice of $u's$ and $v's$ (consider e.g. $v_k = \delta_{k,4}$ and $u_j = \delta_{j,3}$)

$$\implies A_{kj}^* = A_{jk} \implies \mathsf{A} = (\mathsf{A}^*)^\mathsf{T}$$

This means that $\mathsf{A} = \widetilde{\mathsf{A}}$. A matrix with this property is called **self-adjoint**.

We now expand the discussion from matrices to Hilbert spaces. In particular, we are interested in studying symmetric and self-adjoint operators in function spaces.

Given a Hilbert space $\mathcal{H}$ the operator $A \in \mathcal{H}$ is symmetric if it can can be moved from one side to the other of the scalar product without changing it:

$$(u|Av) = (Au|v)$$

The more common name for a symmetric operator is **Hermitian operator**. For a Hermitian operator $\mathcal{A}$ and $u = v$,

$$(u|Au) = (Au|u) = (u|Au)^*$$

Thus, $(u|\mathcal{A}u)$ is a real number. If $(u|Au) \geq 0, \forall u$, then $\mathcal{A}$ is positive semi-definite.

Propositions:
Suppose $\mathcal{A}$ is Hermitian in Hilbert space $\mathcal{H}$. Then all the following applies:

1. All eigenvalues of $\mathcal{A}$ are real.
2. Given two eigenvalues $\lambda_1 \neq \lambda_2$, the corresponding eigenvectors u_1, u_2 are orthogonal.
3. If $\mathcal{A}$ is positive (semi-)definite, all eigenvalues are > 0 (or ≥ 0)

Proof:

1. Assume $Au = \lambda u, u \neq 0$. Then

$$(u|\mathcal{A}u) = (\mathcal{A}u|u) \Longrightarrow (u|\lambda u) = (\lambda u|u) \Longrightarrow \lambda(u|u) = \lambda^*(u|u)$$

   Since $(u|u) > 0$, it must be that $\lambda = \lambda^*$, and therefore λ must be real.

2. Consider

$$\mathcal{A}u_1 = \lambda_1 u_1,\ \mathcal{A}u_2 = \lambda_2 u_2$$

   with $u_1, u_2 \neq 0, \lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$(\mathcal{A}u_1|u_2) = (u_1|\mathcal{A}u_2) \Longrightarrow \lambda_1(u_1|u_2) = \lambda_2(u_1|u_2)$$

   Since $\lambda_1 \neq \lambda_2 \Rightarrow (u_1|u_2) = 0$

3. If $\mathcal{A}$ is positive definite,

$$0 < (u|\mathcal{A}u) = \lambda(u|u)$$

   Because $(u|u) > 0$ for $u \neq 0$, it must be that $\lambda > 0$ (or $\lambda \geq 0$ if $\mathcal{A}$ is semi-positive definite).

EXAMPLE: Consider the integral operator $\mathcal{K} : L_2(I) \longrightarrow L_2(I)$,

$$\mathcal{K}u(x) = \int_I K(x,y)u(y)dy,$$

For an Hermitian operator we get the condition

$$(u|\mathcal{K}v) = (\mathcal{K}u|v)$$

$$\int_I u^*(x)(\mathcal{K}v)(x)dx = \int_I ((\mathcal{K}u)(y))^* v(y)dy$$

$$\Longleftrightarrow \int_I u^*(x)dx \int_I K(x,y)v(y)dy = \int_I (\int_I (K(y,x)u(x)dx)^* v(y)dy$$

$$\Longleftrightarrow \int_I \int_I K(x,y)u^*(x)v(y)dxdy = \int_I \int_I K^*(y,x)u^*(x)v(y)dxdy$$

We choose the following $u(x)$ and $v(y)$:

$$u(x) = \begin{cases} \frac{1}{h} & \text{if } |x - x_0| < h \\ 0 & \text{otherwise} \end{cases}$$

$$y(x) = \begin{cases} \frac{1}{h} & \text{if } |y - y_0| < h \\ 0 & \text{otherwise} \end{cases}$$

By letting $h \to 0$, we find that

$$K(y,x) = K^*(x,y).$$

That an integral operator is symmetric thus provides a condition on its kernel that is analogous to the case where a symmetric matrix have the elements $A_{ij} = A^*_{ji}$

EXAMPLE: Consider the expression

$$\frac{1}{i}D = \frac{1}{i}\frac{d}{dx}, \text{ in } [0, 2\pi],$$

If $u, v \in \mathcal{C}^1(I)$, then

$$\left(u|\frac{1}{i}Dv\right) = \int_0^{2\pi} u^*(x)\frac{v'(x)}{i}dx =$$

$$= \frac{1}{i}\left[u^*(x)v(x)\right]_0^{2\pi} + \int_0^{2\pi}\left(\frac{u'(x)}{i}\right)^* v(x)dx =$$

$$= \frac{1}{i}\left[u^*(x)v(x)\right]_0^{2\pi} + \int_0^{2\pi}\left(\frac{1}{i}Du|v\right)dx.$$

For $(u|\frac{1}{i}Dv) = (\frac{1}{i}Du|v)$ to be true, the expression $[u^*(x)v(x)]_0^{2\pi}$ must be zero. This is the case if u and v are 2π-periodic. The differential operator

$$\mathcal{A} = \frac{1}{i}\frac{d}{dx}$$

is thus an Hermitian operator on $C^1([0, 2\pi])$ for 2π-periodic functions. The eigenvalues and eigenvectors of $\mathcal{A}$ are found by solving the differential equation

$$\mathcal{A}u = \lambda u \Longleftrightarrow u' = i\lambda u$$

with the boundary conditions $u(0) = u(2\pi)$. The general solution to this equation is:

$$u(x) = Ce^{i\lambda x}.$$

Because of the periodic boundary conditions $u(0) = u(2\pi)$,

$$1 = e^{i\lambda 2\pi} \Longleftrightarrow \lambda \text{ is an integer.}$$

Thus, the operator $\mathcal{A}$ has the eigenfunctions e^{ikx} with eigenvalues $k = 0, \pm 1, \pm 2, ...$ We have previously determined this to be an orthogonal basis.

EXAMPLE: By the same token, it can be shown that the differential operator

$$-D^2 = -\frac{d^2}{dx^2}, \ 0 \leq x \leq \pi$$

is Hermitian for functions $u(0) = v(0) = 0, u(\pi) = v(\pi) = 0$. We define the operator $\mathcal{A}$ so that

$$\mathcal{A} = -D^2, \ D_{\mathcal{A}} = \{u \in \mathcal{C}^2([0,\pi])|u(0) = u(\pi) = 0\}.$$

Furthermore,

$$(u|\mathcal{A}u) = (u'|u') = \int_0^{\pi} |u'(x)|^2 dx \geq 0,$$

so the operator $\mathcal{A}$ is *positive semi-definite*. In order to find the eigenvalues and eigenfunctions of $\mathcal{A}$, we must solve the equation

$$\mathcal{A}u = \lambda u \Longleftrightarrow -u'' = \lambda u,$$

with boundary conditions $u(0) = u(\pi) = 0$. This differential equation has the solution

$$\lambda_k = k^2,\ \varphi_k(x) = \sin(kx),\ k = 1, 2, ...$$

The operator $\mathcal{A}$ has an orthogonal basis consisting of the eigenfunctions $\sin(kx)$.

9 Sturm-Liouville operators

An important class of self-adjoint operators are **Sturm-Liouville ordinary differential operators**.

Sturm-Liouville operators arise in many areas of physics and applied mathematics. Here, we will consider the case of ordinary differential equations (ODE). Sturm-Liouville operators also occur in partial differential equations (PDE) but we will not consider these.

ODEs involve n-derivatives, i.e finding a solution requires one or more integration. To uniquely specify the solution we need as many boundary conditions as derivatives. ODEs can be linear, for example:

$$\frac{d^2\phi}{dx^2} + m^2\phi = 0.$$

ODEs can also be non-linear, for example:

$$\frac{d^2\phi}{dx^2} + m^2\sin(\phi) = 0.$$

We will only focus on linear ODEs. A linear ODE can be written on the form

$$\sum_{n=1}^{N} a_n(x)\frac{d^n\phi}{dx^n}(x) = a_0(x).$$

Here,

- all $a_n(x)$ are supposed to be known.
- the largest order derivative decides the order of the equation.
- the equation is homogeneous if $a_0(x) = 0$.
- non-uniqueness is related to lack of specification of suitable boundary conditions. (For an N-order linear ODE, we usually require N boundary conditions.)

A **Sturm-Liouville operator** is a differential operator that can be written on the form

$$a_2(x)\frac{d^2\phi}{dx^2} + a_1(x)\frac{d\phi}{dx} + a_0(x)\phi = \lambda u(x)\phi. \tag{12}$$

We assume that

- $a_0(x), a_1(x), a_2(x)$ and $u(x)$ are real and non-zero for $x \in [a, b]$.
- the function is "well-behaved", i.e. the derivatives exist up to the necessary order.

- the solutions are expected to satisfy boundary conditions.
- the equation is an eigenvalue equation, i.e. λ is *not* a parameter. Rather, it is an eigenvalue and finding λ is part of finding the solution to the Sturm-Liouville problem.

An equivalent way to write (12) is

$$-\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi = \lambda w(x)\phi. \tag{13}$$

The two different ways of expressing the Sturm-Liouville operator can be used to find the solution. First, divide (12) by a_2:

$$\frac{d^2\phi}{dx^2} + \frac{a_1}{a_2}\frac{d\phi}{dx} + \frac{a_0}{a_2} = \lambda\frac{u}{a_2}\phi. \tag{14}$$

Then, divide (13) by $-p(x)$:

$$\frac{d^2\phi}{dx^2} + \frac{1}{p}\frac{dp}{dx}\frac{d\phi}{dx} - \frac{q}{p}\phi = -\lambda\frac{w}{p}\phi, \tag{15}$$

By comparing (14) and (15) we find the equations

$$\frac{1}{p(x)}\frac{dp}{dx} = \frac{a_1(x)}{a_2(x)}$$

$$\frac{q(x)}{p(x)} = -\frac{a_0(x)}{a_2(x)}$$

$$\frac{w(x)}{p(x)} = -\frac{u(x)}{a_2(x)}$$

These equations allow us to move from one expression to the other (see the example below).

The Schrödinger equation is a special case of the Sturm-Liouville problem.

Orthogonal polynomials come from solutions to Sturm-Liouville problems with special boundary conditions. Furthermore, the weight function $w(x)$ in (13) determines the weight function in the scalar product of the solutions.

Example: (Hermite's differential operator.) Consider

$$y'' - 2xy' + \lambda y = 0,\ x \in (-\infty, \infty).$$

Rewrite this as

$$y'' - 2xy' = -\lambda y.$$

After multiplying with e^{-x^2}, this can be written as

$$-(e^{x^2}y')' = \lambda e^{-x^2}y.$$

It is the same as (13), with

$$p(x) = e^{-x^2}, q(x) = 0, w(x) = e^{-x^2}.$$

This is also the same as equation (12), with

$$a_2 = 1, a_1 = -2x, a_0 = 0, u(x) = -1.$$

Thus the relationships

$$\frac{w(x)}{p(x)} = -\frac{u(x)}{a_2(x)}, \quad \frac{1}{p(x)}\frac{dp}{dx} = \frac{a_1(x)}{a_2(x)}$$

are fulfilled.

Consider $\lambda = 2n, n = 0, 1, ...$ One can show that when $\lambda = 2n$, $H_n(x)$ are solutions, and that these make up a complete, orthonormal system in $L_2(-\infty, \infty)$ with the weight function $w(x) = e^{-x^2}$ (verify this!). In particular, the function $z(x) = e^{-x^2} H_n(x)$ satisfies

$$z'' + (2n + 1 - x^2)z = 0,$$

i.e. the Schrödinger equation for the harmonic oscillator.