Hand-ins

Problem 1
Let $I = [0,1]$. Show with the help of the definition that the operator

$$\mathcal{A}u = -\left(u'' + \frac{2}{r}u'\right) = -\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = -\frac{1}{r}\frac{d^2}{dr^2}(ru)$$

$$D_{\mathcal{A}} = \{u \in \mathcal{C}^2[0,1] \,|\mathrm{u\,bounded\,close\,to\,0\,and}\,u(1) = 0\}$$

is symmetric and positive definite in the scalar product

$$(u|v) = \int_0^1 u(r)\,v(r)\,r^2 dr.$$

Determine all eigenvalues and eigenfunctions to $\mathcal{A}$ (Hint: introduce $g(r) = ru(r)$). These are the radial eigenfunctions to the Laplace operator in a sphere with homogeneous Dirichlet boundary conditions.

Problem 2
Let $f_n(x) = x^n$. Are $\{f_n\}_0^\infty$ orthogonal in
a) $L_2([0,1])$?
b) $L_2([-1,1])$?
c)Start from $\{1, x, x^2\}$ and construct an orthogonal set $\{\phi_1, \phi_2, \phi_3\}$ in $L_2([0,1])$
.

Problem 3
Approximate $\sin(x)$ by expanding it in the orthogonal set of functions constructed in problem 3c. Plot the approximations using $\{\phi_1\}$, $\{\phi_1, \phi_2\}$ and $\{\phi_1, \phi_2, \phi_3\}$ as expansion functions and compare the results with the original function ($\sin(x)$)
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Problem 4
a) Use Rodrigues' formula,

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2 - 1\right)^l$$

to calculate the Legendre polynomials $P_0(x)$, $P_1(x)$ and $P_2(x)$.

b) Calculate the associate Legendre polynomials $P_l^m(x)$ for $l = 1$ and $m = 0, 1$. Use the relation

$$\frac{\partial^m}{\partial x^m}P_n(x) = \left(1 - x^2\right)^{-m/2} P_n^m(x)$$

c) Use the result in b) to write up the spherical harmonics $Y_l^m(\theta, \varphi)$ for $l = 1$, $m = 0, 1$. (See page 155 in Gunnar Ohlen's book)
d) Optional bonus question, can you also work out the case $m = -1$? (see e.g. Wikipedia)