Math excercises for Hilbert Spaces

Problem 1 (Linear space)
Determine which of the following sets which, provided with the usual algebraic rules, are linear spaces in $\mathbb{R}$. Specify in the applicable cases a basis, and the dimension of the space. Are there any changes if $\mathbb{R}$ is changed to $\mathbb{C}$?

a): $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 0\}$

b): $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 1\}$

c): {Real $m \times n$ matrices A}

d): {Real $m \times n$ matrices A with det A=0}

e): {Real $m \times n$ matrices A with A=A^T}

f): {Real polynomials of degree n}

g): {Real polynomials of degree $\leq n$ with $p(0) = 1$}

h): {Real polynomials of degree $\leq n$ with $p(1) = 0$}

Problem 2 (Linear space)
Determine which of the following sets that are linear spaces.

a): $\{f \in \mathcal{C}[0,1] \,|\, f(0) + f(1) = 0\}$

b): $\{f \in \mathcal{C}^1(\mathbb{R}) \,|\, f(0) = 0, f'(0) = 1\}$

c): $\{f \in \mathcal{C}^2(\mathbb{R}) \,|\, f'' - f = 0\}$

d): $\{f \in \mathcal{C}[0,1] \,|\, \int_0^1 f(x)\,dx = 0\}$

Specify, in the cases it is possible, the dimension of the space.

Problem 3 (Scalar product and norm)
Let

$$f(x) = 1 + ix,\; g(x) = 2 + ix^2,\; 0 \leq x \leq 1$$

and calculate $(f|g)$ and $(g|f)$, with the scalar product from $L_2[0,1]$.

Problem 4 (Scalar product and norm)
Argue why the following expressions are **not** scalar products in the space of real valued, continous functions in $[0,1]$.

a): $\langle f|g \rangle = \int_0^1 f^2(x)\, g^2(x)\, dx$

b): $\langle f|g\rangle = \int_0^1 f(x)\, g(1-x)\, dx$

Problem 5 (Scalar product and norm)
Verify that, in the fourdimensional linear space of all real 2×2-matrices, the matrices

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ F_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ F_4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

constitute a basis.

For a 2×2-matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the trace is defined as

$$tr\,\mathbf{A} = a_{11} + a_{22}.$$

a): Check that $(\mathbf{A}|\mathbf{B}) = tr\left(\mathbf{A}^T\mathbf{B}\right)$defines a scalar product for real 2×2-matrices.

b): Orthonormalize the matrices F_1, F_2, F_3, F_4 using the scalar product above.

Problem 6 (Projections)
Determine the function $f(t)$, $0 \leq t \leq 1$, with

$$\int_0^1 |f(t)|^2\, dt = 1$$

for which the integral

$$\int_0^1 f(t)\, e^{-t} dt$$

is maximal.

Problem 7 (Projections)
Let u and v in $L_2(0,1)$ be given by

$$u(x) = 1,\ 0 \leq x \leq 1$$
$$v(x) = x,\ 0 \leq x \leq 1$$

Calculate the projection of v on u, and vice versa.

Problem 8 (Gram-Schmidt's orthogonalization method)
In $L_2(0,\infty)$, use the two functions $f(x) = e^{-x}$ and $g(x) = xe^{-x}$ to produce two mutually orthonormal new functions. (Hint: Use the Gram-Schmidt procedure, where integration is on the interval $[0,\infty]$)

Problem 9 (Gram-Schmidt's orthogonalization method)
Expand $f(x) = x^2 e^{-x/2}$ in the Laguerre's basis $f^{(n)}(x) = e^{-x/2} L_n(x)$, where L_n is the n-th Laguerre's polynomial.

Problem 10 (Convergence in norm)
Let $\{p_n\}_0^\infty$ be real orthogonal polynomials in $L_2(I, w)$, deg $p_n = n$ where I is an interval and w is the weight in the scalar product. Show that if $n \geq m$, then

$$p_n(x) p_m(x) = \sum_{n-m}^{n+m} c_k p_k(x)$$

for suitable choices of numbers c_k,which also depends on m and n.

Problem 11 (Operators in Hilbert space)
Let $\mathcal{A}$ be a positiv definite operator on $\mathcal{H}$ i.e. $\mathcal{A}$ is symmetric and $(u|\mathcal{A}u) \geq 0$ for all $u \in D_A$. Show that all eigenvalues of $\mathcal{A}$ are ≥ 0.

Problem 12 (Symmetric operators)
Let $I = [0, a]$, $a > 0$. Show, with the help of the definition , that

$$\mathcal{A} = -\frac{d^2}{dx^2}, \ D_A = \left\{u \in \mathcal{C}^2(I) \, | u(0) = u(a) = 0\right\}$$

is a symmetric and positive definite operator in $L_2(I)$. Determine its eigenvalues and eigenfunctions.

Problem 13 (Symmetric operators)
A matrix U is said to be unitary if $U^\dagger = U^{-1}$ or, equivalently, $U^\dagger U = 1$, where $\left(U^\dagger\right)_{ij} = U_{ji}^*$. It is given the matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & \alpha \end{pmatrix}.$$

a): Determine α such that A is a unitary matrix. (Hint: Use the definitions above)

b): Show that $|\det U| = 1$.

c): Consider that matrix defined as $B = \left(A^\dagger\right)^2 A^3$; determine the trace and the determinant of B. (Hint: Use the property of unitary matrices to shorten the calculation).

Problem 14
Consider two operators A and B, such that their commutator $[A, B] = 1$. Express the commutators $\left[A, B^2\right], \left[A, B^3\right]$ and $\left[A, B^4\right]$ only in terms of B.(use the Jacobi identity $[A, BC] = [A, B]\, C + B\,[A, C]$). On the basis of the previous results, what do you expect the value of $[A, B^n]$, $n > 0$ to be ? Confirm your intuition by induction.

Hint: The commutator is defined as $[A, B] = AB - BA$. If $[A, B] = 0$ one says that A and B commute. In such a case it does not matter in what order we apply the operators on a function since then it follows that $ABf(x) = BAf(x)$.

Problem 15

Find at least one pair of functions $a(x), b(x)$ such that $[a(x) D, b(x) D] = D$, where $D = \frac{d}{dx}$.

Hint: To check if a relation $[A, B] = C$ is true one uses test functions. Thus one checks if $[A, B] f = Cf$ is true for all choices of test functions f.

Answers

Problem 1

a) linear space, dimension: 2

b) not a linear space

c) linear space, dimension: $n \times m$

d) not a linear space

e) linear space, dimension: $\left(n^2 + n\right)/2$

f) not a linear space in case $n \geq 1$

g) not a linear space

h) linear space, dimension: n

No changes occur if we pass from $\mathbb{R}$ to $\mathbb{C}$.

Problem 2

a): linear space, dimension: ∞

b): not a linear space

c): linear space, dimension: 2

d): linear space, dimension: ∞

Problem 3

$(f|g) = \frac{9}{4} - \frac{2}{3}i$ and $(g|f) = \frac{9}{4} + \frac{2}{3}i$

Problem 4

a) non linearity

b) the norm is not > 0 if $f \neq 0$.

Problem 5

To verify if the set of matrices is a basis use the definition of basis.
a) Use the definition of scalar product.
b)

$$E_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} E_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} E_4 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Problem 6

$f(t) = \left[\frac{2}{1-e^{-2}}\right]^{1/2} e^{-t}.$

Problem7

$P_u v = \frac{1}{2}$ and $P_v u = \frac{3}{2}x.$

Problem 8

$\psi_1(x) = \sqrt{2} f(x)$ and $\psi_2(x) = 2\sqrt{2}\left(x - \frac{1}{2}\right) f(x).$

Problem 9

$x^2 = 2L_2 - 4L_1 + 2L_0$

Problem 10

Hint: we know that

$$P_n(x) P_m(x) = \sum_{k=0}^{n+m} c_k p_k(x)$$

where $c_k = \frac{(p_k|p_n p_m)}{(p_k|p_k)}$.

Problem 11

Hint: use the definition of eigenvalue.

Problem 12

Hint: calculate the scalar product and use the boundary conditions. To find the eigennfunctions solve the differential equation obtained following the definition of eigenfunctions.

Problem 13

a) $\alpha = -i$

b) Direct calculation

c) Determinant $=1$

Problem 14

$[A, B^n] = nB^{n-1}$

Problem 15

$a(x) = 1,\ b(x) = x.$