EXAMINATION IN CHAOS

2007-01-11, 8–13

Aids: TEFYMA (or other comparable table) and pocket calculator.

Complete solution required for each task. The examination consists of one theoretical part and one part with problems, each giving 12 points as a maximum. To pass, a reasonable distribution of points is required, apart from a certain minimum score.

THEORETICAL PART.

1. a) Determine the Liapunov exponent if the 'first return map' ('avbildningen för första återkomst') is defined as

$$f(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1. \end{cases}$$

   (1p)

   b) What is meant by *universality* ('universialitet') in connection with one-dimensional maps? (1p)

   c) Consider the dripping faucet ('droppande kranen') in the chaotic region. How is it possible to determine if a 'first return map' exists. Explain briefly the difference between chaotic behaviour and stochastic behaviour! (2p)

2. a) Define a 'self-similar' fractal. (1p)

   b) Describe the Cantor set and define the maps which generate this fractal. Determine its fractal dimension. (2p)

   c) Give a definition of the 'Hausdorff dimension'.

3. Consider Arnold's cat map,

$$\begin{cases} x_{n+1} = x_n + y_n \\ y_{n+1} = x_n + 2y_n \end{cases}$$

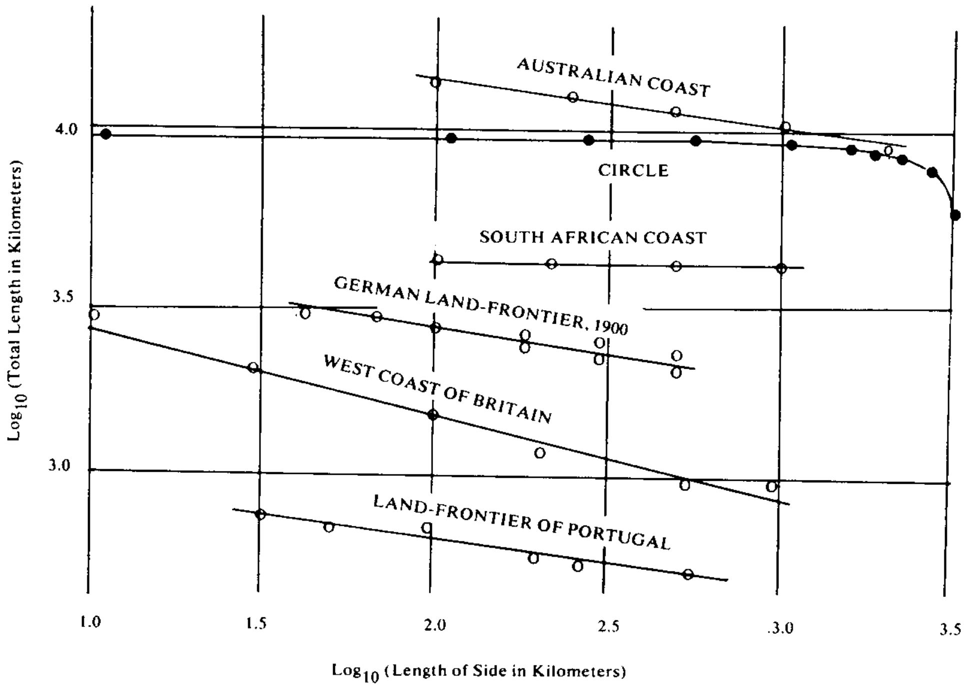
   a) Show that it is area conserving, both (i) analytically and (ii) geometrically by considering the image of a unit square. (2p)

   b) Determine the Liapunov exponents. (2p)

PROBLEMS

4. A one-dimensional iteration is given by $x_{n+1} = rx_n(1 - x_n^2)$. Which values can the parameter r take, if the interval [0,1] is to be invariant? Determine the fixed points and decide for which value of r that the bifurcation from a one-periodic to a two-periodic attractor takes place.

5. The figure on next page shows the lengths of different coast lines and national frontiers when measured with yard sticks of different lengths. Use this figure to estimate the fractal dimension of the 'German land-frontier, 1900'! (3p)



6. An oscillator similar to Duffing's oscillator is described by the differential equation

$$\ddot{x} + 0.08\,\dot{x} + x^5 = A\cos(t)$$

   a) Rewrite the system using phase space variables ('standard form') and show that the system is dissipative. (1p)
   b) Is it possible for the system to have a strange attractor? Motivate! (1p)
   c) For some fixed values of the parameters, one Liapunov exponent is $\lambda_2 = -0.12$. Which is the value of the other Liapunov exponent? Is the system chaotic? (1p)

7. A particle with mass m moves in two dimensions under the influence of the potential

$$V(x, y) = \frac{4}{3}x^3 - \frac{1}{3}y^3$$

   The Hamiltonian will be a function of the coordiantes x and y, and the corresponding momenta, p_x and p_y.

   a) Write down the Hamiltonian (1p).
   b) Write down Hamilton's equations of motion (1p)
   c) Is the system integrable? (1p)