

# Computer exercise for the chaos course

## The Duffing oscillator

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In this exercise you will simulate the non-linear Duffing oscillator

$$\ddot{x} + k\dot{x} + x^3 = A \sin t \quad (1)$$

for different values of the parameters  $k$  and  $A$ .

The exercise can be done on any computer with matlab installed. If you do not have an account for the computer room at physics department (all LTH student should have one), ask us, and we will find a solution. Download the file `duff.zip` from the course homepage, then unpack all files into the same directory and run the matlab script `duffing.m`. Hopefully, the program is self-instructive, but a help text is available via the help menu item. If you have any questions, ask us. Write down brief answers to the following questions, and put your solutions in the box called “kaos in” outside the secretary office.

### 1 Universality

For  $k = 0.05$ , describe qualitatively the sequence of bifurcations as  $A$  grows from 5 to 7.5. Look out for period doubling cascades, chaotic regions, and periodic windows. Write a list of  $A$ -values for which the different dynamics occur. You don't have to follow period doubling cascades more than a few steps. Do you find any similarities or differences with the bifurcations in the logistic map?

(The quickest way to reach an attractor and determine its periodicity is to look at the Poincaré section. The zoom button can be used to resolve nearby points.)

*Optional:* Estimate  $\delta$  for one period doubling cascade. Do you think the Feigenbaum number  $\delta = 4.669$  applies for the Duffing oscillator?

### 2 Basins of attraction

For  $k = 0.05$  and  $A = 0.86$ , first simulate with the initial conditions  $x(0) = 1$ ,  $\dot{x}(0) = 0$ , and then with  $x(0) = 0$ ,  $\dot{x}(0) = 1$ . Does the system reach the same attractor in either case? If not, what is the qualitative difference? Try some other initial conditions, and see what happens. What do you think figure 1 shows?

### 3 The Kaplan-Yorke hypothesis

For the pairs of parameters  $(k, A) \in \{(0.05, 7.5), (0.3, 8), (0.3, 9)\}$ , we have chaotic dynamics. For each pair, estimate the (greatest) Lyapunov exponent  $\lambda_1$  of the flow. If  $d(t)$  is the distance in phase space between two arbitrarily chosen nearby trajectories, then

$$d(t) \approx d(0)e^{\lambda_1 t} \quad (2)$$

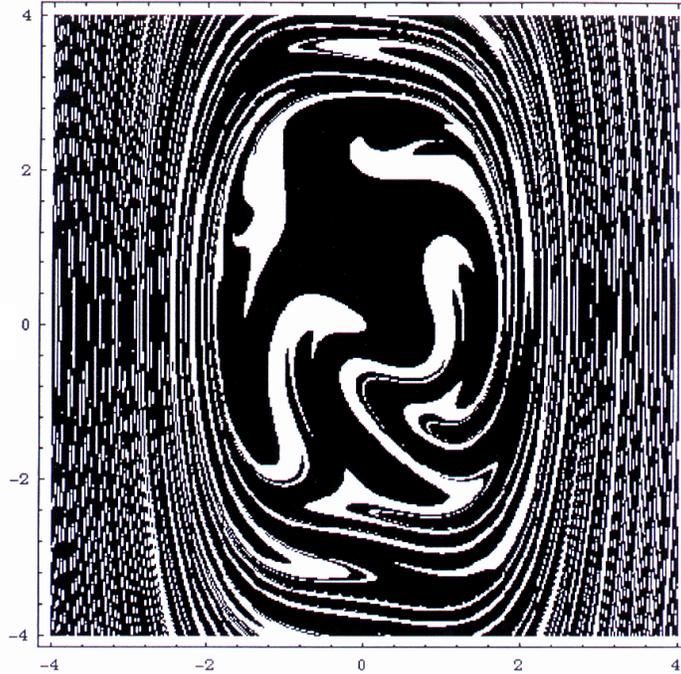


Figure 1: Figure from the master thesis by Niclas Forsberg. The  $x$  and  $y$  axes show  $x$  and  $\dot{x}$

if  $t$  is quite large and  $d(t)$  stays small.

Use the "SIC" display mode, which shows  $\ln d(t)$  as a function of  $t$ . To extract  $\lambda_1$ , estimate a mean slope of the graph before  $\ln d$  saturates. After that, the trajectories do not separate exponentially, and no information about  $\lambda_1$  can be gained. Note that time in the diagram is given in units of the driving force period ( $= 2\pi$ ).

Calculate  $\lambda_2$  from the relation  $\lambda_1 + \lambda_2 = -k$  (see compendium). Finally, calculate the attractor dimension according to Kaplan and Yorke:

$$D_A = 1 - \frac{\lambda_1}{\lambda_2}. \quad (3)$$

Which of the three attractors looks the most two-dimensional, and which the most one-dimensional? Does the Kaplan-Yorke hypothesis seem to hold qualitatively?