Chaos and *Chaos with project*, ht04.

Solutions to selected problems, chapter 3

3.3 Definition:

$$H_s = \lim_{\delta \to 0} \sum_i \mid U_i \mid^s$$

D_H is the s for which H_s goes from ∞ to 0.
We choose the sets $U_i = U_i(n)$ as the straight lines occurring after n iterations. We have $U_i(n) = U_j(n)$ for arbitrary i and j and $\delta = \max_I(\mid U_i \mid) \to 0$ when $n \to \infty$. Thus:

$$H_s = \lim_{n \to \infty} N(n) \mid U_i(n) \mid^s$$

where $N(n)$ is the number of lines after iteration n.

$$N(n) = 5^n, \qquad \mid U_i(n) \mid = k3^{-n},$$

where k is an arbitrary scaling constant. (k is often chosen equal 1, i.e. $\mid U_i(n) \mid = 3^{-n}$)

$$H_s = k^s \lim_{n \to \infty} 5^n \left(\frac{1}{3^n}\right)^s = k^s \lim_{n \to \infty} \left(\frac{5^n}{3^{sn}}\right) = k^s \lim_{n \to \infty} \left(\frac{5}{3^s}\right)^n$$

H_s jumps from ∞ to 0 when $\frac{5}{3^s} = 1$, i.e.

$$5 = 3^{D_H} \qquad \Rightarrow \qquad D_H = \frac{\log 5}{\log 3}$$

3.7 a) Definition of box dimension: for the set F:

$$D_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-log\delta}$$

From the figure, it is easy to see that

$$N(n) = 4^n, \qquad \delta(n) = 3^{-n}$$

i.e. the *length* scale is reduced by a factor 3 in each step. We now obtain:

$$D_B = \frac{\log 4^n}{-\log 3^{-n}} = \frac{\log 4}{\log 3} \approx 1.262$$

b) We use eq. (3.9):

$$\sum_{i=1}^{m} c_i^D = 1$$

In the present case, $c_i = 1/3$ for all four maps $m = 4$. Thus

$$\sum_{i=1}^{m} c_i^D = 1 \quad \Leftrightarrow \quad 4\left(\frac{1}{3}\right) D = 1 \quad \Leftrightarrow \quad D = \frac{\log 4}{\log 3}$$

The formula is applicable because the fractal is self-similar and the different maps do not overlap.

3.11 We should find the Julia set of

$$z(j+1) = z(j)^2 + 4z(j) + 2$$

To find the Julia set from general definitions is difficult so we try to use the derivations in the compendium where the Julia set was studied for

$$z(j+1) = z(j)^2 + c$$

Thus we try to rewrite the present equation so that the right side is the sum of 'the square of a complex number' and a 'constant':

$$z(j+1) = (z(j)+2)^2 - 2$$

Now we should have the same complex number $(z+2)$ on the left side

$$z(j+1) + 2 = (z(j)+2)^2 - 2 + 2 = (z(j)+2)^2$$

and it turns out that the constant vanishes. Thus we can use what was already learned from the z^2 function whose Julia set is the unit circle centered around the origin. With $z \to (z+2)$, the Julia set is a unit circle centered around $z = -2$.

If you use similar methods, it is reasonably straightforward to solve also problem 3.10.