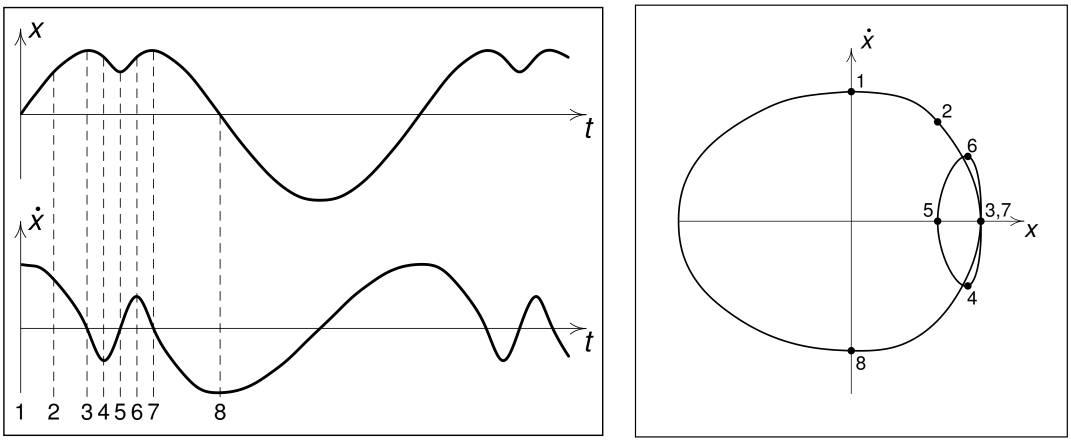
Solutions to selected problems, chapter 4

4.2 With the position x varying as in the upper curve in the panel to the left, we get the velocity $\dot{x}$ as the slope of this curve which comes out as in the lower curve. The $x(t)$ curve drawn here is not equivalent to that in the compendium but it has the same general features, and in the same way, the $\dot{x}(t)$ curve is drawn to show the typical features but not the details.



With $x(t)$ and $\dot{x}(t)$ as in the figure to the left, the trajectory in the $(x, \dot{x})$ plane will come out as in the figure to the right. A few corresponding points in the left and right hand figures are indicated by numbers $1, 2, \ldots, 8$.

The curve crosses itself so it cannot be autonomous.

4.5 The equations for the pendelum in system form are (eq. 4.11):

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{g}{l}\sin x_1 \end{cases}$$

where $x_1 = \phi$ and $x_2 = \dot{\phi}$. The Jacobian is thus obtained as (cf. 4.35):

$$\begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & 0 \end{pmatrix}$$

The two fixed point are $x_1 = 0, x_2 = 0$ and $x_1 = \pi, x_2 = 0$, where the latter corresponds to the upper position. For these values of x_1 and x_2, the Jacobian equals

$$\begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}$$

and the eigenvalues are easily obtained as $h_{1,2} = \pm\sqrt{g/l}$. The eigenvector $\mathbf{v} = (v_1, v_2)$, corresponding to $h_1 = \sqrt{g/l}$, is calculated from,

$$\begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{\frac{g}{l}}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

corresponding to two identical equations $v_1 = \sqrt{g/l}\, v_2$ which means that the normalized eigenvector can be written

$$\mathbf{v} = \left(\sqrt{l}, \sqrt{g}\right) / \sqrt{l+g}$$

Analogous calculations for the other eigenvalue gives the eigenvector $\mathbf{u}$ as

$$\mathbf{u} = \left(\sqrt{l}, -\sqrt{g}\right) / \sqrt{l+g}$$

Inserting the eigenvalues into eq. (4.29), we get

$$\delta\mathbf{x}(\Delta t) = \varepsilon_1 \exp\left(\sqrt{g/l}\,\Delta t\right)\mathbf{v} + \varepsilon_2 \exp\left(-\sqrt{g/l}\,\Delta t\right)\mathbf{u}$$

with the eigenvectors $\mathbf{v}$ and $\mathbf{u}$ given above. It is evident that $\delta\mathbf{x}(\Delta t)$ diverges for large Δt because of the positive eigenvalue $\sqrt{g/l}$.

4.6 Lorenz' equations

$$\begin{cases} \dot{X} = -aX + aY & = \quad f_1(X,Y,Z) \\ \dot{Y} = rX - Y - XZ & = \quad f_2(X,Y,Z) \\ \dot{Z} = XY - bZ & = \quad f_3(X,Y,Z) \end{cases}$$

The origin is a fixed point because $\dot{X} = \dot{Y} = \dot{Z} = 0$ for $X = Y = Z = 0$. In order to determine the stability, we calculate the Jacobian:

$$D\mathbf{f}(\mathbf{X}) = \begin{pmatrix} -a & a & 0 \\ r-Z & -1 & -X \\ Y & X & -b \end{pmatrix}$$

which simplifies to

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} -a & a & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

for $\mathbf{X} = (X,Y,Z) = (0,0,0)$. The matrix split into one 1×1 and another 2×2 matrix which do not couple. Thus two of the eigenvalues $h_{1,2}$ are solutions to the equation

$$\begin{vmatrix} -a-h & a \\ r & -1-h \end{vmatrix} = 0$$

i.e. $(a+h)(1+h) - ar = 0$. It is thus straightforward to obtain:

$$h_{1,2} = -\left(\frac{a+1}{2}\right) \pm \sqrt{\left(\frac{a-1}{2}\right)^2 + ar}$$

If the eigenvalues are real, the fixed point is stable if they are all smaller than 0. This condition is fulfilled if

$$\left(\frac{a+1}{2}\right)^2 > \left(\frac{a-1}{2}\right)^2 + ar$$

i.e. $r < 1$. If $r \ll 0$, h_1 and h_2 becomes imaginary but then, their real part is negative which means that the fixed point remains stable. Consequently, the fixed point at the origin is stable if

Answer: $r < 1$.