Solutions to selected problems, chapter 5

5.4
- The Lie derivative of the Lorenz system was calculated in sect. 4.4.6. It is negative for the standard choice of the parameters. Thus, the system is dissipative.
- The chemical reaction:

$$\begin{cases} \dot{c_A} = -k_1 c_A \cdot c_B + k_2 c_C \\ \dot{c_B} = -k_1 c_A \cdot c_B + k_2 c_C \\ \dot{c_C} = k_1 c_A \cdot c_B - k_2 c_C \end{cases}$$

  The Lie derivative

$$div(\dot{\mathbf{x}}) = \frac{\partial}{\partial c_A}(\dot{c_A}) + \frac{\partial}{\partial c_B}(\dot{c_B}) + \frac{\partial}{\partial c_C}(\dot{c_C}) = -k_1 c_B - k_1 c_A - k_2 < 0$$

  The expression above is smaller than zero because all parameters are positive. Thus, the system is dissipative.
- The electric circuit. It is given in standard form in eq. (4.12):

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\frac{R}{L}x_2 - \frac{1}{L}f(x_1, x_2) + \frac{A}{L}\sin\omega x_3 \\ \dot{x_3} = 1 \end{cases}$$

  The Lie derivative: $\mathrm{div}(\dot{\mathbf{x}}) = 0 - R/L - (1/L)\frac{\partial f}{\partial x_2} + 0$. The voltage across a diode increases with current, i.e. $\frac{\partial f}{\partial x_2} > 0 \Rightarrow div(\dot{\mathbf{x}}) < 0$, i.e. the system is dissipative.
  Note that the system can also be expressed in non-autonomous form:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\frac{R}{L}x_2 - \frac{1}{L}f(x_1, x_2) + \frac{A}{L}\sin\omega t \end{cases}$$

  but it leads to the same value for the Lie derivative

$$div(\dot{\mathbf{x}}) = \frac{\partial}{\partial x_1}(\dot{x_1}) + \frac{\partial}{\partial x_2}(\dot{x_2})$$

5.5 Equation:

$$\ddot{x} + 2\dot{x} + 2x = 2\cos(2t) - \sin(2t) \tag{1}$$

a) We first consider the homogeneous equation

$$\ddot{x} + 2\dot{x} + 2x = 0 \tag{2}$$

and solve it by making the ansatz $x_h = \exp(\lambda t)$ which is inserted in eq. 2. This leads to an equation for λ:

$$\lambda^2 + 2\lambda + 2 = 0$$

which has the solution $\lambda = -1 \pm i$. We thus obtain a solution to the homogeneous equation:

$$\begin{aligned} x_h(t) &= A\exp(-t + it) + B\exp(-t - it) \\ &= \exp(-t)\left(A(\cos t + i\sin t) + B(\cos t - i\sin t)\right) = \\ &= \exp(-t)(C\cos t + D\sin t) \end{aligned}$$

where the constants $C = A + B$ and $D = i(A - B)$ are real because $x(t)$ must be real (alternatively, the solution can be written only using a sine or a cosine function but with a phase, e.g. $x(t) = \exp(-t)F\cos(t + \delta)$ where $C = F\cos\delta$ and $D = -F\sin\delta$).

We also need to find *one* solution to the full equation, a so-called particular solution. We make the ansatz: $x_p(t) = \alpha\sin(\beta t + \varphi)$. When inserted in the original equation, we obtain,

$$-\alpha\beta^2\sin(\beta t + \varphi) + 2\alpha\beta\cos(\beta t + \varphi) + 2\beta\sin(\beta t + \varphi) = 2\cos(2t) - \sin(2t)$$

or

$$2\alpha\beta\cos(\beta t + \varphi) - (\alpha\beta^2 - 2\alpha)\sin(\beta t + \varphi) = 2\cos(2t) - \sin(2t)$$

The equality is fulfilled if $\alpha\beta = 1$, $(\alpha\beta^2 - 2\alpha) = 1$ and $\varphi = 0$,

$$\alpha = \frac{1}{2}, \quad \beta = 2, \quad \varphi = 0$$

General solution:

$$x(t) = x_h(t) + x_p(t) = \exp(-t)(C\cos t + D\sin t)\,\frac{1}{2}\sin(2t)$$

b) see ‘answers’ in compendium

c) For large t, $x(t) = \frac{1}{2}\sin(2t)$ and consequently $\dot{x}(t) = \cos(2t)$. This is thus an attractor which corresponds to an ellipse in the $(x, \dot{x})$-plane.

d) We consider the attractor at $x = 0$ which corresponds to $2t = n\pi$ and thus $\dot{x}(t) = 1$ or $\dot{x}(t) = -1$. The requirement that $\dot{x}(t) > 0$ means that $\dot{x}(t) = 1$ in the Poincaré section, i.e. it will show points approaching $\dot{x}(t) = 1$.

5.10 The simplified Lorenz’ equations:

$$\begin{cases} \dot{X} = -X + Y \\ \dot{Y} = X - Y - XZ \\ \dot{Z} = XY - bZ \end{cases}$$

Velocity $\mathbf{V} = (\dot{X}, \dot{Y}, \dot{Z})$. We get

$$\begin{aligned} \mathbf{V}\cdot\mathbf{R} &= (-X + Y, X - Y - XZ, XY - bZ)\cdot(X, Y, Z) \\ &= -X^2 + XY + XY - Y^2 - XZY + XYZ - bZ^2 \\ &= -X^2 + 2XY - Y^2 - bZ^2 = -(X - Y)^2 - bZ^2 \end{aligned}$$

Assuming that b is positive (standard value), this expression is negative, i.e. $\mathbf{V}\cdot\mathbf{R} < 0$ which means that a trajectory will always approach the origin (see figure).