Solutions to problems; EXAMINATION IN CHAOS

2005-10-21, 14–19

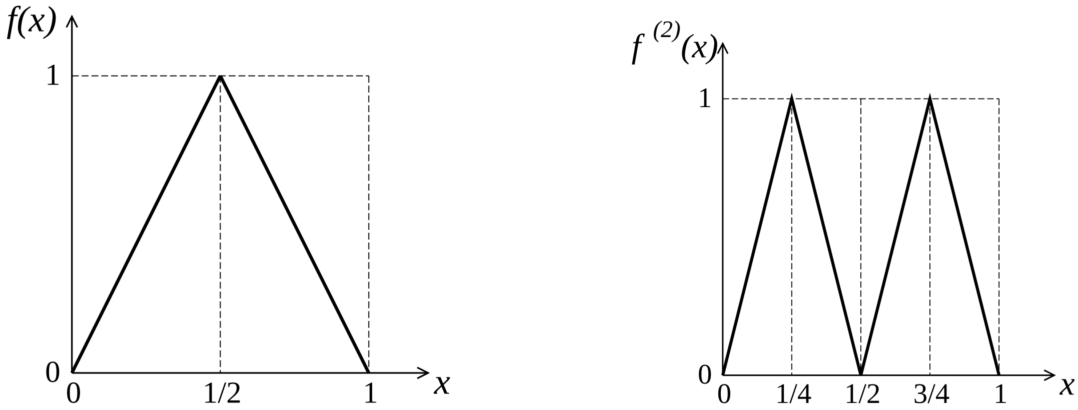
4. The fixed points x^* are determined from $x^* = f(x^*)$. It is then evident that $x^* = 0$ is a fixed point for all a. From the figure of $f(x)$, you conclude that there is another fixed point for $1 \leq a \leq 2$ which is determined from

$$x^* = a(1 - x^*) \qquad \Rightarrow \qquad x^* = \frac{a}{1+a}$$

The Lyapunov exponent is determined from mean value of $\ln |f'(x(j))|$ for an iteration

$$\lambda = \frac{1}{N} \sum_{j=0}^{N-1} \ln |f'(x(j))|$$

in the limit when $N \to \infty$. In the present case, $|f'(x)| = a$ for all x (except $x = 1/2$) which leads to $\lambda = \ln(a)$.



$f(x)$ and $f^{(2)}(x)$ for $a = 2$ are shown above. If one notes that also $f^{(2)}(x)$ must be built from straight lines, it is easy to find $f^{(2)}(x)$ if one calculates it for a few x-values.

If you want to derive $f^{(2)}(x)$ more formally, you have to divide the interval into four parts:

* $0 \leq x \leq 1/4$: In this interval, $f(x) = 2x$ will fall in the interval $[0, 0.5]$ so you should apply $f(x) = 2x$ once more, $f^{(2)}(x) = 2(2x) = 4x$.
* $1/4 \leq x \leq 1/2$: $f(x) = 2x$ will fall in the interval $[0.5, 1.0]$ so you must apply $f(x) = 2(1-x)$ in the second step, $f^{(2)}(x) = 2(1-(2x)) = 2(1-2x)$.
* $1/2 \leq x \leq 3/4$: In this interval, $f(x) = 2(1-x)$ and the result fall in the interval $[0.5, 1.0]$ so you must apply $f(x) = 2(1-x)$ once more, $f^{(2)}(x) = 2(1-(2(1-x))) = 2(2x-1)$.
* $3/4 \leq x \leq 1$: $f(x) = 2(1-x)$ falls in the interval $[0, 0.5]$ so you must apply $f(x) = 2x$ in the second step, $f^{(2)}(x) = 2 * (2(1-x)) = 4(1-x)$.

5. We use the formula

$$\sum_{i=1}^{n} (c_i)^D = 1$$

The first fractal is formed from one map with the length scale $1/4$ and another map with the length scale $1/2$. Thus

$$\left(\frac{1}{4}\right)^D + \left(\frac{1}{2}\right)^D = 1 \qquad \Rightarrow \qquad \left(\left(\frac{1}{2}\right)^D\right)^2 + \left(\frac{1}{2}\right)^D = 1$$

With $x = \left(\frac{1}{2}\right)^D$, you get the equation $x^2 + x - 1 = 0$ which has the solutions

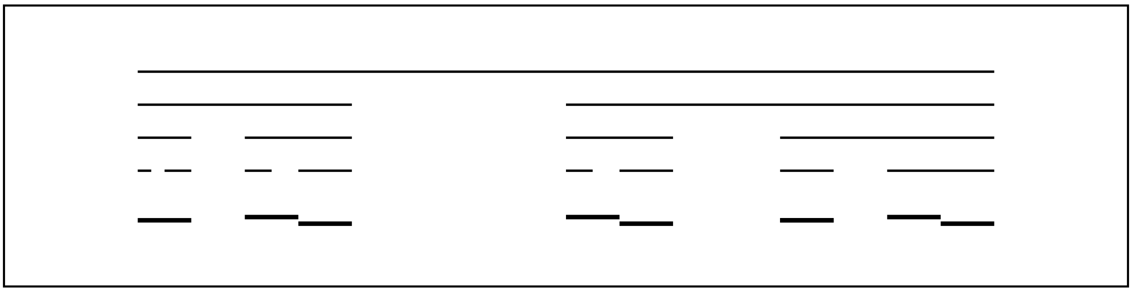
$$x = -\frac{1}{2} \pm \sqrt{1/4 + 1} \quad \Rightarrow \quad \left(\frac{1}{2}\right)^D = \sqrt{1.25} - 0.5$$

where we have chosen the positive solution because $\left(\frac{1}{2}\right)^D$ is positive. Thus $D = -\ln\left(\sqrt{1.25} - 0.5\right)/\ln 2 \approx 0.694$.

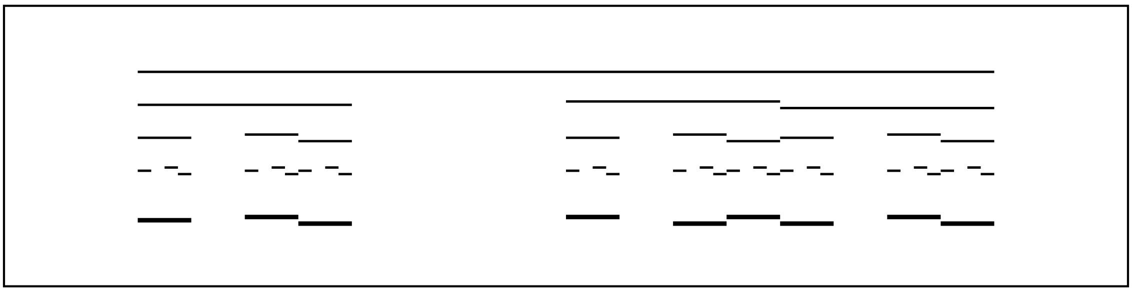
The second fractal is formed from three maps with the length scale $1/4$. Thus

$$3\left(\frac{1}{4}\right)^D = 1 \quad \Rightarrow \quad D = \frac{\ln(1/3)}{\ln(1/4)} = \frac{\ln 3}{\ln 4} \approx 0.792$$

c) If the first fractal is iterated 3 steps, it can be covered by 8 lines of length $L/16$ as shown by thick lines below. Note that you would have needed 9 lines if you have iterated only 2 steps. Furthermore, because one line is longer than the measuring unit after three iterations ($L/8$ compared with $L/16$), you must convince yourself that you will not need fewer than 8 lines to cover the fractal after a fourth iteration.



For the second fractal, you would only need to iterate two steps but three iterations are illustrated below. You need 9 lines to cover the fractal.



The fact that you need more lines to cover the second fractal is consistent with the fact that it has a larger dimension than the first fractal.

6. The two equations can be written, $\dot{x} = f_1(x, y), \dot{y} = f_2(x, y)$. A fixed point corresponds to $\dot{x} = \dot{y} = 0$. It is then trivial to insert $(x, y) = \left(\frac{c}{d}, \frac{a}{b}\right)$ and show that $f_1(c/d, a/b) = 0$ and $f_2(c/d, a/b) = 0$.

   The stability is determined from the eigenvalues of the Jacobian, which are calculated from the Jacobian:

$$D\mathbf{f}(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

   With f_1 and f_2 as given in the text:

$$D\mathbf{f}(x, y) = \begin{pmatrix} rc\left(-\frac{1}{x^2}\right) + a - by & -bx \\ dy & -ra\left(-\frac{1}{y^2}\right) - c + dx \end{pmatrix}$$

and thus

$$D\mathbf{f}\left(\frac{c}{d},\frac{a}{b}\right)=\begin{pmatrix} -rc\left(\frac{d^2}{c^2}\right)+a-b(a/b) & -b(c/d) \\ d(a/b) & ra\left(\frac{b^2}{a^2}\right)-c+d(c/d) \end{pmatrix}=\begin{pmatrix} -r\left(\frac{d^2}{c}\right) & -bc/d \\ da/b & r\left(\frac{b^2}{a}\right) \end{pmatrix}$$

The eigenvalues $h_{1,2}$ are now determined from the equation

$$\begin{vmatrix} -r\left(\frac{d^2}{c}\right)-h & -bc/d \\ da/b & r\left(\frac{b^2}{a}\right)-h \end{vmatrix}=0$$

i.e.

$$\left(r\left(\frac{d^2}{c}\right)+h\right)\left(r\left(\frac{b^2}{a}\right)-h\right)-(bc/d)\,(da/b)=0$$

This equation has the solution:

$$h_{1,2}=\frac{r}{2}\frac{b^2c-ad^2}{ac}\pm\sqrt{\frac{r^2}{4}\left(\frac{b^2c-ad^2}{ac}\right)^2+\frac{r^2b^2d^2}{ac}-ac}$$

With higher orders of r neglected:

$$h_{1,2}=\frac{r}{2}\frac{b^2c-ad^2}{ac}\pm i\sqrt{ac}$$

In this limit, the eigenvalues are complex and the stability is determined from $\mathrm{Re}(h_{1,2})$. Because $b^2c-ad^2>0$, $\mathrm{Re}(h_{1,2})<0$ for $r<0$ which means that the equilibrium is stable while $r>0$ leads to $\mathrm{Re}(h_{1,2})>0$ corresponding to an unstable equilibrium.

7. When writing the map in standard form, you just put in $x(j+1)$ according to the first equation into the second equation. Thus

$$\begin{cases} x(j+1)=2x(j)+y(j)^\alpha=f_1(x(j),y(j)) \\ y(j+1)=2x(j)+y(j)^\alpha+b\,(x\,(j)+y\,(j))=f_2(x(j),y(j)) \end{cases}$$

where the second equation can be rewritten as

$$y(j+1)=(2+b)x(j)+y(j)^\alpha+by\,(j)$$

The map is area-conserving if the determinant of the Jacobian is equal to 1, i.e.

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}=\begin{vmatrix} 2 & \alpha y^{\alpha-1} \\ 2+b & b+\alpha y^{\alpha-1} \end{vmatrix}=1$$

This equation leads to $2b-b\alpha y^{\alpha-1}=1$ which can be fulfilled for all values of y either if $\alpha=1$, leading to the equation $2b-b=1$ and thus $b=1$, or for $\alpha=0$ leading to $2b=1$ and thus $b=1/2$. Note also that the general requirement for area conservation is rather $|\det(\mathbf{f}(x,y))|=1$ and with $\det(\mathbf{f}(x,y))$ negative instead, you get a sign change for b.
c) We will consider the (non-trivial) values $\alpha=1$ and $b=1$:

$$\begin{cases} x(j+1)=2x(j)+y(j) \\ y(j+1)=3x(j)+2y(j) \end{cases}$$

The eigenvalues are obtained from the equation

$$\begin{vmatrix} 2-\tilde{h} & 1 \\ 3 & 2-\tilde{h} \end{vmatrix}=0$$

which leads to $\tilde{h}_{1,2} = 2 \pm \sqrt{3}$. The eigenvalues are thus independent of x and y which means that the Lyapunov exponents must be the same for all iterations. They are obtained as

$$\begin{aligned} \lambda_1 &= \ln(2+\sqrt{3}) \approx 1.317 \\ \lambda_2 &= \ln(2-\sqrt{3}) \approx -1.317 \end{aligned}$$

With $\alpha = 0$ and $b = 1/2$, $\tilde{h}_1 = 2$ and $\tilde{h}_2 = 1/2$ and thus

$$\begin{aligned} \lambda_1 &= \ln 2 \approx 0.693 \\ \lambda_2 &= \ln(1/2) \approx -0.693 \end{aligned}$$

Also for these values of α and b, the eigenvalues are independent of x and y so the Lyapunov exponents are the same for all iterations also in this case.

Alternatively, from the fact that the functions are linear for both values of b and α, you can conclude that the Lyapunov exponents must be independent of the initial values but you are still asked to calculate their values.