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Exercises to Electrodynamics, Week 5

Homework to be handed in on Dec. 8 in Yousefs mailbox

Exercise 1: *Energy flow in rectangular wave guide*

The lowest TE mode in an ideal rectangular wave guide with $L_x = a > L_y = b$ reads (Jackson 8.46)

$$k^2 = \mu\epsilon\omega^2 - (\pi/a)^2 \qquad \tilde{H}_z = H_0 \cos(\pi x/a)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}$$
$$\tilde{E}_y = \mathrm{i}\omega a\mu H_0/\pi \sin(\pi x/a)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t} \qquad \tilde{H}_x = -\mathrm{i}kaH_0/\pi \sin(\pi x/a)\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}$$

a) Evaluate the energy flow P

b) Evaluate the energy loss per length $-\mathrm{d}P/\mathrm{d}z$ for a non-ideal waveguide with conductivity σ.
Hint: The energy loss per area at the boundary point $\mathbf{r}_s$ reads (Sec. 8.1 of Jackson)

$$\frac{\mathrm{d}P_{\text{loss}}}{\mathrm{d}A} = |\tilde{\mathbf{H}}_{\|}(\mathbf{r}_s)|^2\sqrt{\frac{\mu_c\omega}{8\sigma}}$$

c) On which length scale $1/\alpha$ drops the energy flow by a factor of e? (this is not a simple expression)

Exercise 2: *Energy in a cavity*

Consider a cavity based on the waveguide of exercise 1 covered with an ideal metal at $z = 0$ and $z = d > a, b$. Then the lowest mode is given by $\epsilon\mu\omega_0^2 = (\pi/a)^2 + (\pi/d)^2$ with

$$\tilde{H}_z = H_0 \cos(\pi x/a)\sin(\pi z/d)\mathrm{e}^{-\mathrm{i}\omega_0 t}$$
$$\tilde{H}_x = -H_0 a/d \sin(\pi x/a)\cos(\pi z/d)\mathrm{e}^{-\mathrm{i}\omega_0 t}$$
$$\tilde{E}_y = \mathrm{i}\omega a\mu H_0/\pi \sin(\pi x/a)\sin(\pi z/d)\mathrm{e}^{-\mathrm{i}\omega_0 t}$$

Evaluate the energy $U_E(t) = \int \mathrm{d}^3r\, \epsilon\mathbf{E}^2(t)/2$ and $U_B(t) = \int \mathrm{d}^3r\, \mu\mathbf{H}^2(t)/2$ contained in the electric and magnetic field, respectively, and show that the sum $U(t)$ is constant in time.
Hint: Consider the real fields $\mathbf{H}(t) = \Re\{\tilde{\mathbf{H}}(t)\}$, $\mathbf{E}(t) = \Re\{\tilde{\mathbf{E}}(t)\}$

Exercise 3 (Homework): *TM mode in rectangular wave guide*

Evaluate the lowest TM mode spectrum of the wave guide from exercise 1. Determine the dispersion relation $\omega(k)$ and the electric and magnetic fields.

Exercise 4 (Homework): *TEM mode in coaxial cable*

a) Evaluate the electric and magnetic fields of the TEM mode in an ideal coaxial cable with inner radius R_i and outer radius R_o. (You may use that $\mathbf{E}_r(x, y) = E_r(r)\mathbf{e}_r$ in cylindrical coordinates due to the boundary conditions.)

b) Evaluate the energy flow $P(z, t)$ carried by the fields.

c) Evaluate the current $I(z, t)$ in the central wire and the bias drop $U(z, t) = \int_{R_i}^{R_o} \mathrm{d}r E_r(r)$. Interpret the ratio $U(z, t)/I(z, t)$! Compare the the product $U(z, t)I(z, t)$ with b)!

Exercise 5 (Homework): *Waveguide absorption*

The optical field of a Quantum Cascade Laser operating at 3 THz is confined between two Cu plates with a distance of 10 μm [B.S. Williams, S. Kumar, Q. Hu, and J.L. Reno, Optics Express **13**, 3331 (2005)]. The waveguide is essentially filled with GaAs ($\epsilon = 13\epsilon_0$). Assume that the waveguide is infinitely wide and the mode shape is TEM (the finite waveguide would exhibit a TM mode). Calculate the waveguide absorption using $\sigma_{\mathrm{Cu}} = 6\times 10^7$A/Vm, which is appropriate for room temperature.