2. Introduction to quantum mechanics

2.1 Linear algebra

Dirac notation

Complex conjugate	z^*
Vector/ket	$\|\psi\rangle$
Dual vector/bra	$\langle\varphi\|$
Inner product/bracket	$\langle\varphi\|\psi\rangle$
Tensor product	$\|\varphi\rangle \otimes \|\psi\rangle \equiv \|\varphi\rangle\|\psi\rangle$
Complex conj. matrix	A^*
Transpose of matrix	A^T
Hermitian conj/ adjoint of matrix	$A^\dagger = (A^T)^*$
Inner product	$\langle\varphi\|A\|\psi\rangle$

Basis, vector representation

For a set of vectors $|v_1\rangle, \ldots, |v_n\rangle$ spanning $\mathbf{C}^n$

$$|v\rangle = \sum_i a_i |v_i\rangle \equiv \begin{bmatrix} a_1 \\ a_2 \\ .. \\ a_n \end{bmatrix}$$

The set $|v_1\rangle, \ldots, |v_n\rangle$ constitutes a *basis* for $\mathbf{C}^n$

Linear operators

A *linear operator* A means

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A(|v_i\rangle)$$

Notation

$$A|v\rangle \equiv A(|v\rangle) \qquad BA|v\rangle \equiv B(A(|v\rangle))$$

Matrix representation

For $|v_1\rangle, \ldots, |v_n\rangle$ spanning $\mathbf{C}^n$, $|w_1\rangle, \ldots, |w_m\rangle$ spanning $\mathbf{C}^m$,

a *matrix representation* of $A : \mathbf{C}^n \to \mathbf{C}^m$ means

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

Numbers A_{ij} form matrix A $\Rightarrow$

Linear operator (basis given) $\Leftrightarrow$ matrix representation

(*to be used interchangeably...*)

Pauli matrices

$$\sigma_0 \equiv I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Inner vector product

A *inner product* on $\mathbf{C}^n$ is

$$\Big((a_1, ..., a_n), (b_1, ..., b_n)\Big) = \sum_{i=1}^{n} a_i^* b_i$$

We use notation

$$\langle w|v\rangle \equiv (|w\rangle, |v\rangle)$$

Hilbert space = inner product space

The vectors $|v\rangle, |w\rangle$ are *orthogonal* if

$$\langle w|v\rangle = 0$$

The *norm* of a vector is

$$\||v\rangle\| = \sqrt{\langle v|v\rangle}$$

An *orthonormal set* of vectors $|i\rangle$ obey

$$\langle j|i\rangle = \delta_{ij}$$

Vector representation

With respect to an orthonormal basis $|i\rangle$ for $\mathbf{C}^n$

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \\ .. \\ a_n \end{bmatrix} \qquad |w\rangle = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{bmatrix}$$

the *inner product* is

$$\langle w|v\rangle = [b_1^*, b_2^*, ..., b_n^*] \begin{bmatrix} a_1 \\ a_2 \\ .. \\ a_n \end{bmatrix}$$

We thus have

$$\langle w| = \sum_i b_i^* \langle i| \equiv [b_1^*, b_2^*, .., b_n^*]$$

(*an orthonormal basis will be used unless otherwise stated*)

Outer vector product

The *outer product*

$$|\varphi\rangle\langle\psi|$$

is a linear operator

$$|\varphi\rangle\langle\psi|\left(|\psi'\rangle\right) = |\varphi\rangle\langle\psi|\psi'\rangle = \langle\psi|\psi'\rangle|\varphi\rangle$$

Cauchy-Schwartz inequality

For two vectors $|v\rangle, |w\rangle$

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2$$

Completeness relation

For vectors $|i\rangle$ forming an orthonormal basis $\langle j|i\rangle = \delta_{ij}$ for $\mathbf{C}^n$

$$\sum_{i=1}^{n} |i\rangle\langle i| = I$$

Eigenvectors and eigenvalues

The *eigenvector* $|v\rangle$ to A obeys

$$A|v\rangle = v|v\rangle$$

with v the *eigenvalue*.

The *diagonal representation* of A is (for diagonalizable A)

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

in terms of eigenvalues λ_i and orthonormal eigenvectors $|i\rangle$ of A

Hermitian operators

The *Hermitian conjugate*/adjoint of A is $A^\dagger$

We have $(AB)^\dagger = B^\dagger A^\dagger$ and $|v\rangle^\dagger = \langle v|$, $(|v\rangle\langle w|)^\dagger = |w\rangle\langle v|$

An *Hermitian operator* obeys

$$A^\dagger = A$$

Projection operator

The operator

$$P = \sum_{i=1}^{m} |i\rangle\langle i|$$

is a *projection* operator $P : \mathbf{C}^n \to \mathbf{C}^m, \; m < n$

Properties

$$P^2 = P \qquad\qquad P^\dagger = P$$

Orthogonal complement $Q = I - P$

Normal operator

An operator A is *normal* if

$$A^\dagger A = A A^\dagger$$

An operator is normal if and only if it is diagonalizable.
An Hermitian operator is normal.

Unitary matrix

A matrix/operator U is *unitary* if

$$U^\dagger U = UU^\dagger = I$$

Positive operator

An operator A is *positive* if

$$\langle v|A|v\rangle \geq 0$$

and real for any vector $|v\rangle$.

Any positive operator is Hermitian $\Rightarrow$
Any positive operator has real, positive eigenvalues
and a *spectral decomposition*

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

in terms of eigenvalues λ_i and orthonormal eigenvectors $|i\rangle$ of A

Tensor product

A *tensor product* between vectors $|v\rangle, |w\rangle$ in $\mathbf{C}^n, \mathbf{C}^m$

$$|v\rangle \otimes |w\rangle \equiv |v\rangle|w\rangle \equiv |vw\rangle$$

is a vector in $\mathbf{C}^{n\times m}$

Example:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{bmatrix}$$

A tensor product between operators/matrices A, B is denoted

$$A \otimes B$$

Operation

$$A \otimes B(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

Properties

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

Matrix representation

Example:

For matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

we have the tensor product

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

Operator functions

For a normal operator A , written in the spectral decomposition

$$A = \sum_a a|a\rangle\langle a|$$

we define the/operator matrix function

$$f(A) = \sum_a f(a)|a\rangle\langle a|$$

Trace

The *trace* of a matrix A is

$$\mathrm{tr}(A) = \sum_i A_{ii}$$

Cyclic property

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$$

Outer product formulation

$$\mathrm{tr}(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$$

Commutators

The *commutator* between two operators/matrices A, B is

$$[A, B] = AB - BA$$

The *anti-commutator* between two operators/matrices A, B is

$$\{A, B\} = AB + BA$$

Matrix decompositions

Polar decomposition: For a linear operator A there exists a unitary operator U and positive operators J, K so that

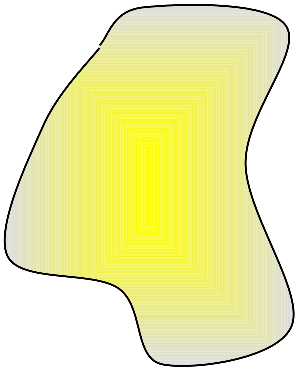
$$A = UJ = KU$$

Singular value decomposition: For a square matrix A there exists unitary matrices U, V and a diagonal matrix D with non-negative elements (singular values), such that

$$A = UDV$$

2.2 Postulates of quantum mechanics

$\mathbf{C}^n \times \mathbf{C}^n$



State space

Postulate 1:

Associated to any isolated physical system is a Hilbert space, known as the *state space* of the system. The system is completely described by its *state vector*, a unit vector in the state space

Definitions/names

A two-level, *qubit* state $|\psi\rangle$ can generally be written as

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

This is a *superposition* of the two basis states $|0\rangle$ and $|1\rangle$, with *amplitudes* a and b

The *normalization condition* gives

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1$$

Evolution

Postulate 2:

The evolution of a quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t is related to the state $|\psi'\rangle$ of the system at time t' by a unitary operator U as

$$|\psi'\rangle = U|\psi\rangle$$

Postulate 2':

The evolution of state $|\psi\rangle$ of a quantum system is described by the *Schrödinger equation*

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

where $\hbar$ is Plancks constant and H the Hamiltonian, a Hermitian operator.

Closed system

For a closed system the Hamiltonian H is independent on time and the system state $|\psi(t)\rangle$ is

$$|\psi(t_2)\rangle = \exp\left[\frac{-iH(t_2-t_1)}{\hbar}\right]|\psi(t_1)\rangle = U(t_2,t_1)|\psi(t_1)\rangle$$

where we define the unitary time evolution operator

$$U(t_2,t_1) = \exp\left[\frac{-iH(t_2-t_1)}{\hbar}\right]$$

The Hamiltonian has the *spectral decomposition*

$$H = \sum_E E|E\rangle\langle E|$$

where $|E\rangle$ are the *energy eigenstates* and E the *energy*.

Effective Hamiltonian for open systems

For many open systems we have an effective time dependent Hamiltonian acting on the system $\Rightarrow$
The solution to the Schrödinger equation is non-trivial

Measurement

Projection measurement postulate

A projective measurement is described by an *observable*, M, an Hermitian operator on the state space of the system. The observable has the spectral decomposition

$$M = \sum_m m P_m = \sum_m m|m\rangle\langle m|$$

The possible outcomes correspond to the eigenvalues m of M.

Upon measuring the state $|\psi\rangle$, the probability of getting the result m is given by

$$p(m) = \langle\psi|P_m|\psi\rangle$$

Given that m has occured, the state immediatelly after the measurement is (wavefunction collapse)

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$$

Measurement problem?

General measurement

Postulate 3:

Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*, acting on the state space of the system. The index m refers to the possible measurement outcomes. Upon measuring the state $|\psi\rangle$, the probability of getting the result m is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

and the state after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^\dagger M_m = I$$

Probabilities sum to one

$$\sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = 1$$

Projective vs general measurement

For a projective measurement

$$M_m = P_m \qquad\qquad M_m M_{m'} = \delta_{mm'} M_m$$

The *average* measured value (over an ensemble of states $|\psi\rangle$)

$$\sum_m m p(m) = \sum_m \langle\psi|P_m|\psi\rangle = \langle\psi| \left(\sum_m m P_m \right) |\psi\rangle = \langle\psi|M|\psi\rangle \equiv \langle M\rangle$$

The magnitude of the quantum fluctuations are

$$\langle(\Delta M)^2\rangle = \langle(M - \langle M\rangle)^2\rangle = \langle M^2\rangle - \langle M\rangle^2$$

Derivation: Heisenbergs uncertainty principle is

$$\Delta(C)\Delta(D) \equiv \sqrt{\langle(\Delta C)^2\rangle\langle(\Delta D)^2\rangle} \geq \frac{1}{2}|\langle[C,D]\rangle|$$

Composite systems

Postulate 4:

The state space of a composite system is the tensor product of the state spaces of the component systems.

If we have systems numbered 1 through n, and system i is prepared in state $|\psi_i\rangle$, the state of the total system is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$$

General measurement and projection II

Derivation: Given projection measurements and an *ancilla* system, derive the general measurement principles.

POVM measurement

Common formulation of general measurement postulate.

A measurement is described by measurement operators M_m .
The probability to get the outcome m is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

We define the positive operator

$$E_m = M_m^\dagger M_m$$

which has the properties

$$\sum_m E_m = I \qquad\qquad p(m) = \langle\psi|E_m|\psi\rangle$$

We call E_m the *POVM-elements* and the set $\{E_m\}$ a *POVM.*

Distinghuishing quantum states

Given a single copy of one of two non-orthogonal states $|\psi_1\rangle, |\psi_2\rangle$, it is not possible to determine which state by any measurement.

Derivation: The example with $\{E_1, E_2, E_3\}$

Entanglement of two qubits

A composite state $|\psi\rangle$ of two qubits that can not be written as a tensor product of the states $|a\rangle, |b\rangle$ of the two qubits is *entangled*, that is

$$|\psi\rangle \neq |a\rangle|b\rangle$$

Derivation: Show that the Bell state

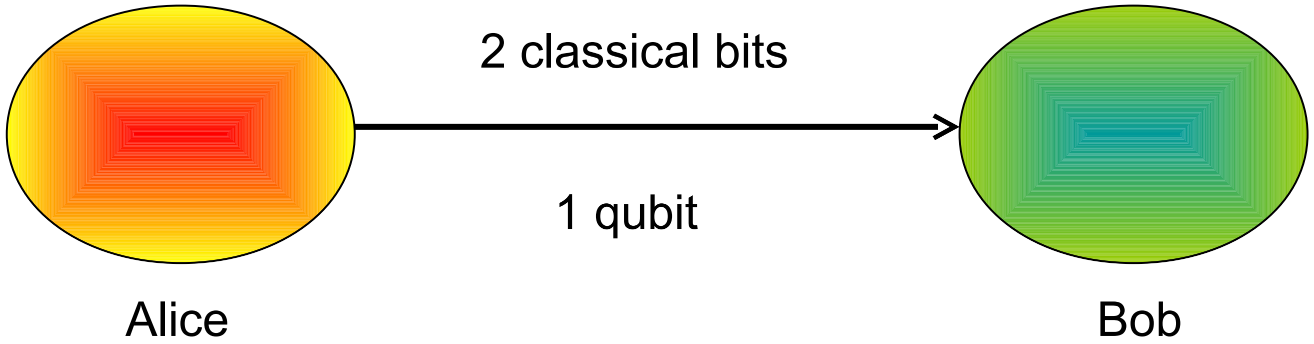
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

is entangled

Entanglement is the "energy" for quantum information processing

2.3 Superdense coding

Suppose that Alice wants to send two bits of classical information to Bob by only sending one qubit. Can she do it?



Derivation: Superdense coding