# 2. Introduction to quantum mechanics

## 2.1 Linear algebra

**Dirac notation**

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Basis, vector representation

For a set of vectors $|v_1\rangle, \ldots, |v_n\rangle$ spanning $\mathbb{C}^n$

$$|v\rangle = \sum_i a_i |v_i\rangle \equiv \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

The set $|v_1\rangle, \ldots, |v_n\rangle$ constitutes a basis for $\mathbb{C}^n$

Linear operators

A linear operator $A$ means

$$A \left( \sum_i a_i |v_i\rangle \right) = \sum_i a_i A(|v_i\rangle)$$

Notation

$$A|v\rangle \equiv A(|v\rangle) \quad \text{and} \quad BA|v\rangle \equiv B(A(|v\rangle))$$
Matrix representation

For $|v_1\rangle, \ldots, |v_n\rangle$ spanning $\mathbb{C}^n$, $|w_1\rangle, \ldots, |w_m\rangle$ spanning $\mathbb{C}^m$, a \textit{matrix representation} of $A : \mathbb{C}^n \rightarrow \mathbb{C}^m$ means

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Numbers $A_{ij}$ form matrix $A$.

Linear operator (basis given) $\iff$ matrix representation

$(\textit{to be used interchangeably} \ldots)$

Pauli matrices

\[
\sigma_0 \equiv I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \quad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \quad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
Inner vector product

A *inner product* on $\mathbb{C}^n$ is

$$\left((a_1, \ldots, a_n), (b_1, \ldots, b_n)\right) = \sum_{i=1}^{n} a_i^* b_i$$

We use notation

$$\langle w | v \rangle \equiv \langle |w\rangle, |v\rangle \rangle$$

*Hilbert space* = inner product space

The vectors $|v\rangle, |w\rangle$ are *orthogonal* if

$$\langle w | v \rangle = 0$$

The *norm* of a vector is

$$||v|| = \sqrt{\langle v | v \rangle}$$

An *orthonormal set* of vectors $|i\rangle$ obey

$$\langle j | i \rangle = \delta_{i,j}$$
Vector representation

With respect to an orthonormal basis $|i\rangle$ for $\mathbb{C}^n$

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \\ . . . \\ a_n \end{bmatrix} \quad \text{and} \quad |w\rangle = \begin{bmatrix} b_1 \\ b_2 \\ . . . \\ b_n \end{bmatrix}$$

the inner product is

$$\langle w|v \rangle = [b_1^*, b_2^*, . . . , b_n^*] \begin{bmatrix} a_1 \\ a_2 \\ . . . \\ a_n \end{bmatrix}$$

We thus have

$$\langle w| = \sum_i b_i^* \langle i| \equiv [b_1^*, b_2^*, . . . , b_n^*]$$

(an orthonormal basis will be used unless otherwise stated)
Outer vector product

The outer product

\[ |\varphi\rangle \langle \psi| \]

is a linear operator

\[ |\varphi\rangle \langle \psi| (|\psi\rangle \langle \psi'|) = |\varphi\rangle \langle \psi |\psi'\rangle = \langle \psi |\psi'\rangle |\varphi\rangle \]

Cauchy-Schwartz inequality

For two vectors \( |v\rangle, |w\rangle \)

\[ \langle v|v\rangle \langle w|w\rangle \geq |\langle v|w\rangle|^2 \]

Completeness relation

For vectors \( |i\rangle \) forming an orthonormal basis \( \langle j|i\rangle = \delta_{ij} \) for \( \mathbb{C}^n \)

\[ \sum_{i=1}^{n} |i\rangle \langle i| = I \]
Eigenvectors and eigenvalues

The eigenvector $|v\rangle$ to $A$ obeys

$$A|v\rangle = v|v\rangle$$

with $v$ the eigenvalue.

The diagonal representation of $A$ is (for diagonalizable $A$)

$$A = \sum_i \lambda_i |i\rangle \langle i|$$

in terms of eigenvalues $\lambda_i$ and orthonormal eigenvectors $|i\rangle$ of $A$.

Hermitian operators

The Hermitian conjugate/adjoint of $A$ is $A^\dagger$

We have $(AB)^\dagger = B^\dagger A^\dagger$ and $|v\rangle^\dagger = \langle v|$, $(|v\rangle \langle w|)^\dagger = |w\rangle \langle v|$.

An Hermitian operator obeys

$$A^\dagger = A$$
Projection operator

The operator

\[ P = \sum_{i=1}^{m} |i\rangle\langle i| \]

is a *projection* operator \( P : \mathbb{C}^n \rightarrow \mathbb{C}^m, \ m < n \)

Properties

\[ P^2 = P \quad P^\dagger = P \]

Orthogonal complement \( Q = I - P \)

Normal operator

An operator \( A \) is *normal* if

\[ A^\dagger A = AA^\dagger \]

An operator is normal if and only if it is diagonalizable.

An Hermitian operator is normal.
Unitary matrix

A matrix/operator $U$ is **unitary** if

$$U^\dagger U = UU^\dagger = I$$

Positive operator

An operator $A$ is **positive** if

$$\langle v | A | v \rangle \geq 0$$

and real for any vector $|v\rangle$.

Any positive operator is Hermitian $\Rightarrow$
Any positive operator has real, positive eigenvalues and a **spectral decomposition**

$$A = \sum_i \lambda_i |i\rangle \langle i|$$

in terms of eigenvalues $\lambda_i$ and orthonormal eigenvectors $|i\rangle$ of $A$. 
Tensor product

A tensor product between vectors $|v\rangle, |w\rangle$ in $\mathbb{C}^n, \mathbb{C}^m$

$|v\rangle \otimes |w\rangle \equiv |v\rangle |w\rangle \equiv |vw\rangle$

is a vector in $\mathbb{C}^{n \times m}$

Example:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1w_1 \\ v_1w_2 \\ v_2w_1 \\ v_2w_2 \end{bmatrix}$$

A tensor product between operators/matrices $A, B$ is denoted

$A \otimes B$

Operation

$A \otimes B(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$
Properties

\[(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger\]

Matrix representation

Example:

For matrices

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\quad B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\]

we have the tensor product

\[
A \otimes B = \begin{bmatrix}
A_{11}B & A_{12}B \\
A_{21}B & A_{22}B
\end{bmatrix}
= \begin{bmatrix}
A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\
A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\
A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\
A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22}
\end{bmatrix}
\]
Operator functions

For a normal operator $A$, written in the spectral decomposition

$$A = \sum_a a |a\rangle \langle a|$$

we define the operator matrix function

$$f(A) = \sum_a f(a) |a\rangle \langle a|$$

Trace

The trace of a matrix $A$ is

$$\text{tr}(A) = \sum_i A_{ii}$$

Cyclic property

$$\text{tr}(ABC) = \text{tr}(CAB) = \text{tr}(BCA)$$

Outer product formulation

$$\text{tr}(A|\psi\rangle \langle \psi|) = \langle \psi|A|\psi\rangle$$
Commutators

The *commutator* between two operators/matrices $A, B$ is

$$[A, B] = AB - BA$$

The *anti-commutator* between two operators/matrices $A, B$ is

$$\{A, B\} = AB + BA$$

Matrix decompositions

*Polar decomposition*: For a linear operator $A$ there exists a unitary operator $U$ and positive operators $J, K$ so that

$$A = UJ = KU$$

*Singular value decomposition*: For a square matrix $A$ there exists unitary matrices $U, V$ and a diagonal matrix $D$ with non-negative elements (singular values), such that

$$A = UDV$$
2.2 Postulates of quantum mechanics

State space

Postulate 1:

Associated to any isolated physical system is a Hilbert space, known as the state space of the system. The system is completely described by its state vector, a unit vector in the state space.

Definitions/names

A two-level, qubit state $|\psi\rangle$ can generally be written as

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

This is a superposition of the two basis states $|0\rangle$ and $|1\rangle$, with amplitudes $a$ and $b$

The normalization condition gives

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1$$
Evolution

Postulate 2:

The evolution of a quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at time $t$ is related to the state $|\psi'\rangle$ of the system at time $t'$ by a unitary operator $U$ as

$$|\psi'\rangle = U|\psi\rangle$$

Postulate 2':

The evolution of state $|\psi\rangle$ of a quantum system is described by the Schrödinger equation

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

where $\hbar$ is Planck's constant and $H$ the Hamiltonian, a Hermitian operator.
Closed system

For a closed system the Hamiltonian $H$ is independent on time and the system state $|\psi(t)\rangle$ is

$$|\psi(t_2)\rangle = \exp \left[ -\frac{iH(t_2 - t_1)}{\hbar} \right] |\psi(t_1)\rangle = U(t_2, t_1)|\psi(t_1)\rangle$$

where we define the unitary time evolution operator

$$U(t_2, t_1) = \exp \left[ -\frac{iH(t_2 - t_1)}{\hbar} \right]$$

The Hamiltonian has the spectral decomposition

$$H = \sum_E E |E\rangle \langle E|$$

where $|E\rangle$ are the energy eigenstates and $E$ the energy.

Effective Hamiltonian for open systems

For many open systems we have an effective time dependent Hamiltonian acting on the system $\Rightarrow$

The solution to the Schrödinger equation is non-trivial
Measurement

Projection measurement postulate

A projective measurement is described by an observable, $M$, an Hermitian operator on the state space of the system. The observable has the spectral decomposition

$$M = \sum_{m} m P_m = \sum_{m} m |m\rangle \langle m|$$

The possible outcomes correspond to the eigenvalues $m$ of $M$.

Upon measuring the state $|\psi\rangle$, the probability of getting the result $m$ is given by

$$p(m) = \langle \psi | P_m | \psi \rangle$$

Given that $m$ has occurred, the state immediately after the measurement is (wavefunction collapse)

$$\frac{P_m |\psi\rangle}{\sqrt{p(m)}}$$

Measurement problem?
Postulate 3:

Quantum measurements are described by a collection \( \{ M_m \} \) of measurement operators, acting on the state space of the system. The index \( m \) refers to the possible measurement outcomes. Upon measuring the state \( |\psi\rangle \), the probability of getting the result \( m \) is given by

\[
p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle
\]

and the state after the measurement is

\[
\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m | \psi \rangle}}
\]

The measurement operators satisfy the completeness relation

\[
\sum_m M_m^\dagger M_m = I
\]

Probabilities sum to one

\[
\sum_m p(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle = 1
\]
Projective vs general measurement

For a projective measurement

\[ M_m = P_m \quad M_m M_{m'} = \delta_{m m'} M_m \]

The average measured value (over an ensemble of states \(|\psi\rangle\))

\[ \sum_m m p(m) = \sum_m \langle \psi | P_m | \psi \rangle = \langle \psi | \left( \sum_m m P_m \right) | \psi \rangle = \langle \psi | M | \psi \rangle \equiv \langle M \rangle \]

The magnitude of the quantum fluctuations are

\[ \langle (\Delta M)^2 \rangle = \langle (M - \langle M \rangle)^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2 \]

**Derivation:** Heisenberg's uncertainty principle is

\[ \Delta(C) \Delta(D) \equiv \sqrt{\langle (\Delta C)^2 \rangle \langle (\Delta D)^2 \rangle} \geq \frac{1}{2} |\langle [C, D] \rangle| \]
Composite systems

**Postulate 4:**

The state space of a composite system is the tensor product of the state spaces of the component systems.

If we have systems numbered 1 through \( n \), and system \( i \) is prepared in state \( |\psi_i\rangle \), the state of the total system is

\[
|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle
\]

General measurement and projection II

**Derivation:** Given projection measurements and an *ancilla* system, derive the general measurement principles.
POVM measurement

Common formulation of general measurement postulate.

A measurement is described by measurement operators $M_m$. The probability to get the outcome $m$ is given by

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$$

We define the positive operator

$$E_m = M_m^\dagger M_m$$

which has the properties

$$\sum_m E_m = I \quad p(m) = \langle \psi | E_m | \psi \rangle$$

We call $E_m$ the POVM-elements and the set $\{E_m\}$ a POVM.

Distinguishing quantum states

Given a single copy of one of two non-orthogonal states $|\psi_1\rangle, |\psi_2\rangle$, it is not possible to determine which state by any measurement.

**Derivation:** The example with $\{E_1, E_2, E_3\}$
Entanglement of two qubits

A composite state $|\psi\rangle$ of two qubits that cannot be written as a tensor product of the states $|a\rangle$, $|b\rangle$ of the two qubits is entangled, that is

$$|\psi\rangle \neq |a\rangle|b\rangle$$

**Derivation:** Show that the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}} [ |00\rangle + |11\rangle ]$$

is entangled

Entanglement is the "energy" for quantum information processing
Suppose that Alice wants to send two bits of classical information to Bob by only sending one qubit. Can she do it?

**Derivation:** Superdense coding