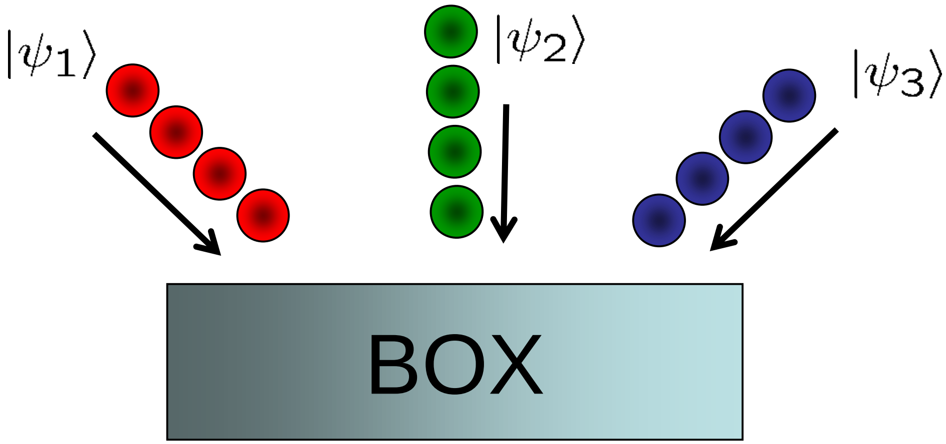
2.4 Density operator/matrix

Ensemble of *pure states* gives a *mixed state*



The *density operator* or *density matrix* ρ for the ensemble or mixture of states $|\psi_i\rangle$ with probabilities p_i is given by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad \sum_i p_i = 1$$

Note: Once mixed, there is due to indistinghuishability of quantum particles not way of ”unmixing”. Example:

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \frac{1}{2}\left[|a\rangle\langle a| + |b\rangle\langle b|\right] \qquad |a/b\rangle = \frac{1}{2}\left[\sqrt{3}|0\rangle \pm |1\rangle\right]$$

Derivation: General qubit density matrix.

Time development

For an individual state $|\psi_i\rangle$, with U the unitary time evolution operator

$$|\psi_i\rangle \to U|\psi_i\rangle \qquad \Rightarrow$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \to \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger$$

Measurement

We perform a general measurement described by M_m. If the system is in state $|\psi_i\rangle$, the (conditional) probability to get m is

$$p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = \mathrm{tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|)$$

The total probability to get m when measuring on ρ is then

$$\sum_i p_i p(m|i) = \sum_i p_i \mathrm{tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) = \mathrm{tr}(M_m^\dagger M_m \rho)$$

Post-measurement state

For an initial state $|\psi_i\rangle$, the state after measuring m is

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$$

The total state after measuring m on ρ is

$$\rho_m = \sum_i \frac{p_i p(m|i)|\psi_i^m\rangle\langle\psi_i^m|}{p(m)}$$

The normlization condition $\mathtt{tr}(\rho_m) = 1$ gives the denominator

$$p(m) = \sum_i p_i p(m|i)$$

Inserting known expressions, this gives

$$\rho_m = \sum_i p_i \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\mathtt{tr}(M_m^\dagger M_m \rho)} = \frac{M_m \rho M_m^\dagger}{\mathtt{tr}(M_m^\dagger M_m \rho)}$$

Composition

If we have systems numbered 1 through n, and system i is in state ρ_i , the state of the total system is

$$\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n$$

General properties

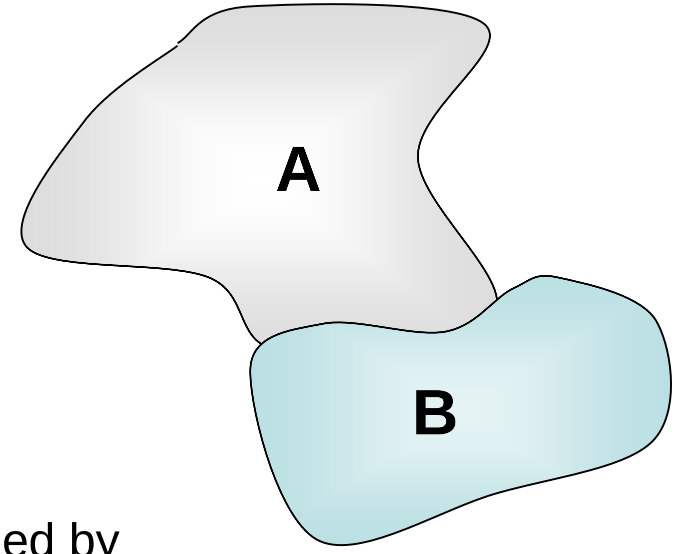
A operator ρ is a density operator if and only if it

1) ρ has trace equal to one.
2) ρ is a positive operator.

Derivation: Exercise 2.71, purity of a state ρ .

Reduced density operator/matrix

The composite, total system AB is in the state ρ_{AB}.



The *reduced density operator* of system A is by definition

$$\rho_A = \text{tr}_B\,(\rho_{AB})$$

where the *partial trace* over B is defined by

$$\text{tr}_B\,(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2|\text{tr}(|b_1\rangle\langle b_2|)$$

with $|a_1\rangle, |a_2\rangle$ $(|b_1\rangle, |b_2\rangle)$ any vectors in A (B), and the linearity property of the trace.

The reduced density operator describes completely all the properties/outcomes of measurements of the system A, given that system B is left unobserved (”tracing out” system B)

Derivation: Properties of reduced density operator.

Derivation: Reduced density matrix for Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$

2.5 Schmidt decomposition and purification

Schmidt decomposition

For a pure state $|\psi\rangle$ in the composite system AB, there exists orthonormal bases $|i_A\rangle$ and $|i_B\rangle$ (*Schmidt bases*) for systems A and B, such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

with λ_i the real, non-negative Schmidt coefficients and

$$\sum_i \lambda_i^2 = 1$$

for $|\psi\rangle$ normalized, $\langle\psi|\psi\rangle = 1$.

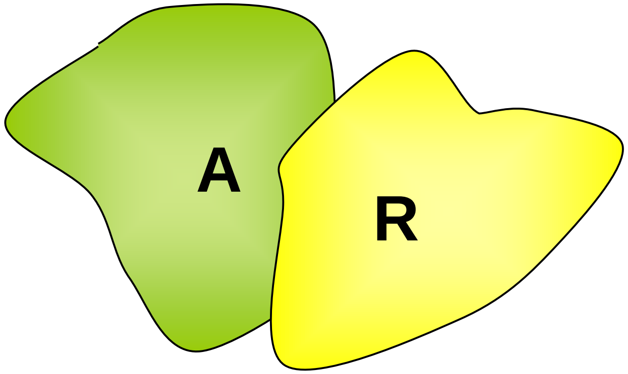
Properties

The reduced density matrices ρ_A, ρ_B have the same eigenvalues

$$\rho_A = \mathrm{tr}_B\,(|\psi\rangle\langle\psi|) = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \qquad \rho_B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

The *Schmidt number* is the number of non-zero Schmidt coefficients.

<u>Purification</u>



Consider a system A in state ρ_A.

It is possible to introduce an additional system R and to define a pure state $|AR\rangle$, such that the reduced density operator for A is

$$\mathrm{tr}_R(|AR\rangle\langle AR|) = \rho_A$$

This is called *purification*.

We can construct $|AR\rangle$ by first noting that we can spectrally decompose

$$\rho_A = \sum_i p_i |i_A\rangle\langle i_A|$$

By taking R to have the same dimensions as A we can define

$$|AR\rangle = \sum_i \sqrt{p_i}|i_A\rangle|i_R\rangle$$

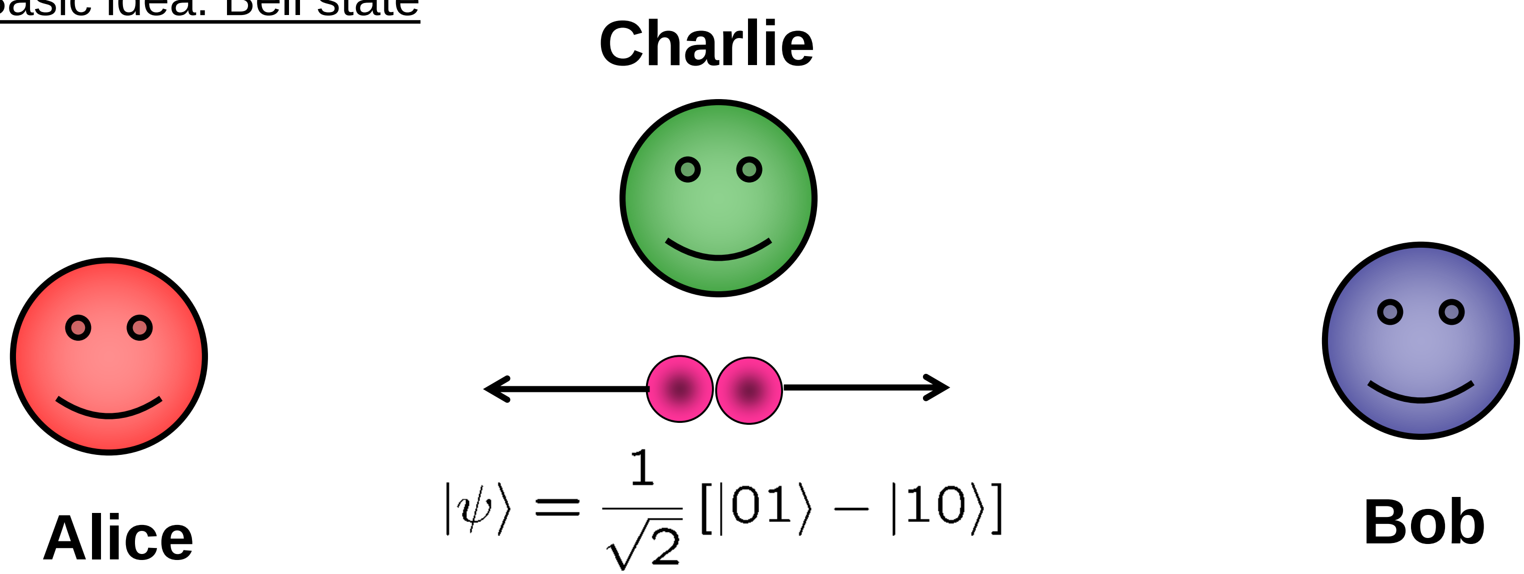
This then gives

$$\mathrm{tr}_R(|AR\rangle\langle AR|) = \sum_{ij} \sqrt{p_i p_j}|i_A\rangle\langle j_A|\mathrm{tr}(|i_R\rangle\langle j_R|) = \sum_i p_i|i_A\rangle\langle i_A| = \rho_A$$

2.6 EPR and the Bell Inequality

- Einstein, Podoldsky, Rosen vs Bohr – local realism vs. Quantum mechanics.
- Bell – experimental test of local realistic theories, Bell Inequality.
- Experiments so far in line with quantum mechanics

Basic idea: Bell state



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$$

1) Charlie prepares two particles/qubits in a Bell state and sends one to Alice and one to Bob.
2) Alice and Bob measure on their particles at the same time.
3) Later, Alice and Bob meet and compare measurement results.

Quantum mechanics

If Alice and Bob measure in the basis $|0\rangle, |1\rangle$, they get a series, e.g.

A $|0\rangle\ |1\rangle\ |0\rangle\ |0\rangle\ |1\rangle\ |0\rangle\ |1\rangle\ |0\rangle$ ....

anti-correlated

B $|1\rangle\ |0\rangle\ |1\rangle\ |1\rangle\ |0\rangle\ |1\rangle\ |0\rangle\ |1\rangle$ ....

If Alice and Bob measure in another basis, $|-\rangle, |+\rangle$ with
$|\pm\rangle = (1/\sqrt{2})[|0\rangle \pm |1\rangle]$ they get another series, e.g.

A $|-\rangle|-\rangle|+\rangle|-\rangle|+\rangle|+\rangle|-\rangle|+\rangle$ ....

anti-correlated

B $|+\rangle|+\rangle|-\rangle|+\rangle|-\rangle|-\rangle|+\rangle|-\rangle$ ....

since the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle] = \frac{1}{\sqrt{2}}[|-+\rangle - |+-\rangle]$$

Important: Before Alice and Bob measure, the particles can not be assigned any particular properties as e.g. $|0\rangle, |+\rangle$

Local realism (not postulated in quantum mechanics)

1) Before (independent on) the measurement, each particle has a specific property for any type of measurement, i.e. $|0\rangle$ if the measurement takes place in the $|0\rangle, |1\rangle$ basis (element of reality).
2) This property is merely revealed by the experiment.
3) The property can not be influenced by any measurement done at another location at the same time (locality assumption)

Local realistic description of the Bell state measurement

- Charlie prepares a set of pairs of classical particles.
- Each particle has a predermined value for the possible experiment, e.g. $|0\rangle, |+\rangle$.
- Alice and Bobs particles have oppposite properties, i.e $|0\rangle, |+\rangle$ for Alice and $|1\rangle, |-\rangle$ for Bob.
- Charlie choses a statistical distribution of the particles (equal weight of all four combination) such that the quantum mechanical measurement result is recovered.

A	$|0\rangle, |+\rangle$	$|1\rangle, |+\rangle$	$|1\rangle, |-\rangle$	$|0\rangle, |-\rangle$	$|1\rangle, |-\rangle$	....
B	$|1\rangle, |-\rangle$	$|0\rangle, |-\rangle$	$|0\rangle, |+\rangle$	$|1\rangle, |+\rangle$	$|0\rangle, |+\rangle$	....

Bell inequality

Bell proposed a scheme for an experimental test of local realistic theories vs quantum mechanics.

Derivation: Bell inequality.