Main Results of Vector Analysis

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1 Repetition: Vector Space

Consider a $d-$dimensional *real vector space* $\mathcal{V}$ with *scalar product* (or inner product) $\mathbf{v}\cdot\mathbf{w}$. Here, the elements of $\mathcal{V}$ are denoted by bold-face letters.

In order to perform calculations, we choose a convenient basis $\mathcal{B}=\{\mathbf{e}_1,\dots\mathbf{e}_d\}$, so that each vector can be uniquely written as $\mathbf{v}=\sum_{i=1}^d v_i\mathbf{e}_i$. This provides a mapping

$$\mathbf{v}\to\begin{pmatrix}v_1\\v_2\\\dots\\v_d\end{pmatrix}$$

which depends on the particular choice of the basis.

For an *orthonormal basis* (ON basis), which satisfies $\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}$, we find the simple rule for scalar product

$$\mathbf{v}\cdot\mathbf{w}=\sum_{i=1}^d v_iw_i=\begin{pmatrix}v_1 & \dots & v_d\end{pmatrix}\begin{pmatrix}w_1\\\dots\\w_d\end{pmatrix}$$

For real three-dimensional vector spaces, one can further define the *cross product* (or outer product) $\mathbf{c}=\mathbf{a}\times\mathbf{b}$
Here, the vector $\mathbf{c}$ is determined by its direction $\mathbf{c}\perp\mathbf{a},\mathbf{b}$ (i.e. $\mathbf{c}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{b}=0$) with *positive orientation* (right-hand rule: thumb= $\mathbf{a}$, index finger= $\mathbf{b}$, middle finger= $\mathbf{c}$) and the length $|\mathbf{c}|=\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2}$ which corresponds to the area $|\mathbf{a}||\mathbf{b}|\sin\angle(\mathbf{a},\mathbf{b})$ of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$. Beware that the cross product is neither commutative nor associative! For an orthonormal basis with positive orientation (i.e. $\mathbf{e}_3=\mathbf{e}_2\times\mathbf{e}_1$ and cyclic) the components of the respective vectors satisfy

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}\qquad\text{or}\qquad c_i=\sum_{kl}\epsilon_{ikl}a_kb_l$$

The *triple scalar product* (only for three-dimensional vector spaces) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ provides the volumes of the parallelepiped formed by the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. It satisfies

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

The *triple cross product* satisfies the important **BAC-CAB** rule

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

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2 Multivariable functions

We consider an affine space of points P, where for a given origin O each point P is uniquely related to a vector $\mathbf{r} = \overrightarrow{OP} \in \mathcal{V}$. Here $\mathcal{V}$ is the associated real vector space. For the elements of P we define *scalar functions* $f(P)$ and *vector functions* $\mathbf{F}(P)$. If we fix the origin of the point space, they can be written as $f(\mathbf{r})$ and. $\mathbf{F}(\mathbf{r})$.

2.1 Systems of coordinates

In order to perform any calculations, we have to express the points by specific coordinates. Choosing an (ON)-basis $\mathcal{B} = \{\mathbf{e}_1, \dots \mathbf{e}_d\}$, we have $\mathbf{r} = \sum_{i=1}^d r_i \mathbf{e}_i$ and we may write the scalar function as $f(\mathbf{r}) = f_{\mathcal{B}}(r_1, \dots r_d)$.

In the standard three-dimensional space, the basis vectors are denoted as $\mathcal{B} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the corresponding *Cartesian coordinates* as x, y, z. For problems with a rotational symmetry around the $z-$axis, *cylindrical coordinates* (ρ, φ, z) are usually a good choice. For systems with spherical symmetry one generally uses spherical coordinates (r, φ, θ), see Tab. 1 on page 11 for a compilation. Thus the identical function $f(\mathbf{r})$ can be represented by different expressions. An example is

$$f(\mathbf{r}) = \frac{1}{|\mathbf{r}|} = \underbrace{\frac{1}{\sqrt{x^2+y^2+z^2}}}_{=f_{\text{Cartesian}}(x,y,z)} = \underbrace{\frac{1}{\sqrt{\rho^2+z^2}}}_{=f_{\text{cylindrical}}(\rho,\varphi,z)} = \underbrace{\frac{1}{r}}_{=f_{\text{spherical}}(r,\varphi,\theta)}$$

2.2 Examples

Elevation in geography: The elevation is a scalar function $h(\mathbf{r})$ of the points on earth's surface. It can be conveniently parameterized by the longitude φ and the latitude $|\theta - \pi/2|$ (so that the poles are at 90° and the equator at 0°)

Current of a fluid: Here $n(\mathbf{r}, t)$ is the *particle density*, so that the number of particles $\mathrm{d}N$ in the volume element $\mathrm{d}V$ around the position $\mathbf{r}$ at time t is given by

$$\mathrm{d}N = n(\mathbf{r}, t) \underbrace{\mathrm{d}V}_{\text{or } \mathrm{d}^3 r}$$

This is a scalar function. The *velocity field* $\mathbf{v}(\mathbf{r}, t)$ denotes the velocity of a test particle following the current. It is a vector function alike the *particle current density* $\mathbf{j}(\mathbf{r}, t) = n(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)$.

3 Multidimensional derivatives

3.1 Partial and total derivative

In general the derivative of a function f with respect to a quantity t is based on the definition of a mapping $t \to f(t)$. Then the derivative is given by the limit $\mathrm{Der}_t f = \lim_{\epsilon \to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon}$. If we consider functions $f(x, y, z, t)$, which depend on more than one variable, one needs to specify how all variables depend on the quantity t with respect to which the differentiation is performed. Thus we need to specify functions $x(t), y(t), z(t)$ and than we can write

$$\mathrm{Der}_t f = \frac{\mathrm{d}f(x, y, z, t)}{\mathrm{d}t} = \lim_{\epsilon \to 0} \frac{f(x(t+\epsilon), y(t+\epsilon), z(t+\epsilon), t+\epsilon) - f(x(t), y(t), z(t), t)}{\epsilon}$$

which is usually called the *total derivative*. Frequently, the functions $x(t), y(t), z(t)$ are not explicitly stated, if the physical problem suggests a natural choice such as the path of a particle in time.

A special choice is that the functions $x(t), y(t), z(t)$ are constants. Then one writes

$$\mathrm{Der}_t f = \frac{\partial f(x,y,z,t)}{\partial t} = \lim_{\epsilon\to 0}\frac{f(x,y,z,t+\epsilon)-f(x,y,z,t)}{\epsilon}$$

which is called *partial derivative*. Note that the partial derivative only exists, if the quantity t is one of the explicit variables of the function. Furthermore the value of the partial derivative may change by coordinate transformations. Thus it is of utmost importance to specify the variables of the function. If there is a natural set of variables, physicists frequently become sloppy here (which sometimes causes misunderstandings).

The total derivative can be conveniently expressed in terms of partial derivatives via

$$\frac{\mathrm{d}f(x,y,z,t)}{\mathrm{d}t} = \frac{\partial f(x,y,z,t)}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \frac{\partial f(x,y,z,t)}{\partial y}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{\partial f(x,y,z,t)}{\partial z}\frac{\mathrm{d}z(t)}{\mathrm{d}t} + \frac{\partial f(x,y,z,t)}{\partial t}$$

and one defines the *total differential* as

$$\mathrm{d}f(x_1,x_2,\dots x_d) = f(x_1+\mathrm{d}x_1, x_2+\mathrm{d}x_2,\dots x_d+\mathrm{d}x_d) = \sum_{i=1}^{d}\frac{\partial f(x_1,x_2,\dots x_d)}{\partial x_i}\mathrm{d}x_i$$

In the same way second derivatives are defined. An important relation is the *symmetry of the second derivatives* (sometimes also referred to as Schwarz's theorem or Clairaut's theorem)

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f(x_1,x_2,\dots x_d)}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f(x_1,x_2,\dots x_d)}{\partial x_i}\right) = \frac{\partial^2 f(x_1,x_2,\dots x_d)}{\partial x_i \partial x_j}$$

provided that the second derivatives are continuous.

3.2 Gradient

Consider the change of a scalar function $f(\mathbf{r})$ due to a change in space $\mathrm{d}\mathbf{r} = \mathbf{e}_x\mathrm{d}x + \mathbf{e}_y\mathrm{d}y + \mathbf{e}_z\mathrm{d}z$ in Cartesian coordinates. We find

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z = \left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}\right) f(\mathbf{r})\cdot\mathrm{d}\mathbf{r}$$

We define the *vector differential operator (del)* which is represented by the symbol ∇ (nabla) by the relation

$$\mathrm{d}f = \nabla f(\mathbf{r})\cdot\mathrm{d}\mathbf{r} \tag{1}$$

In the Cartesian basis it reads

$$\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}$$

The vector function $\mathbf{G}(\mathbf{r}) = \nabla f(\mathbf{r})$ can be interpreted as follows: Consider $f(\mathbf{r})$ in the vicinity of $\mathbf{r}_0$. Then

$$f(\mathbf{r}_0+\mathrm{d}\mathbf{r}) = f(\mathbf{r}_0) + \mathbf{G}(\mathbf{r}_0)\cdot\mathrm{d}\mathbf{r} = f(\mathbf{r}_0) + |\mathbf{G}(\mathbf{r}_0)||\mathrm{d}\mathbf{r}|\cos(\angle(\mathbf{G}(\mathbf{r}_0),\mathrm{d}\mathbf{r}))$$

Thus $\mathbf{G}(\mathbf{r}_0)$ points in the direction of the largest growth of $f(\mathbf{r})$ and the absolute value $|\mathbf{G}(\mathbf{r}_0)|$ provides the value of the largest growth per distance $|\mathrm{d}\mathbf{r}|$. Thus $\mathbf{G}(\mathbf{r}_0)$ is also called the *gradient* of $f(\mathbf{r})$ at position $\mathbf{r}_0$.

The *gradient* of a scalar function

$$\operatorname{grad} f(\mathbf{r}) = \nabla f(\mathbf{r})$$

is a vector function. In Cartesian coordinates it is given by

$$\operatorname{grad} f(\mathbf{r}) \to \begin{pmatrix} \frac{\partial f_x(x,y,z)}{\partial x} \\ \frac{\partial f_x(x,y,z)}{\partial y} \\ \frac{\partial f_x(x,y,z)}{\partial z} \end{pmatrix}$$

Examples: (to work out)

$$\operatorname{grad} |\mathbf{r}-\mathbf{a}| = \frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|}\,, \qquad \operatorname{grad} g(|\mathbf{r}-\mathbf{a}|) = g'(|\mathbf{r}-\mathbf{a}|)\frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|}$$

In particular

$$\operatorname{grad} \frac{1}{|\mathbf{r}-\mathbf{a}|} = -\frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} \tag{2}$$

3.3 Divergence, Curl and Laplace operator

The Del operator can also be applied to vector functions $\mathbf{F}(\mathbf{r})$. There are two different variants.

The *divergence* of a vector function

$$\operatorname{div} \mathbf{F}(\mathbf{r}) = \nabla \cdot \mathbf{F}(\mathbf{r})$$

is a scalar function. In Cartesian coordinates it is given by

$$\operatorname{div} \mathbf{F}(\mathbf{r}) = \frac{\partial F_x(x,y,z)}{\partial x} + \frac{\partial F_y(x,y,z)}{\partial y} + \frac{\partial F_z(x,y,z)}{\partial z}$$

Example: (to work out) $\operatorname{div} \mathbf{r} = 3$

The *curl* of a vector function (only in three dimensions) $\nabla \times \mathbf{F}(\mathbf{r})$ is a new vector function. In Cartesian coordinates

$$\nabla \times \mathbf{F}(\mathbf{r}) \to \begin{pmatrix} \frac{\partial F_z(x,y,z)}{\partial y} - \frac{\partial F_y(x,y,z)}{\partial z} \\ \frac{\partial F_x(x,y,z)}{\partial z} - \frac{\partial F_z(x,y,z)}{\partial x} \\ \frac{\partial F_y(x,y,z)}{\partial x} - \frac{\partial F_x(x,y,z)}{\partial y} \end{pmatrix}$$

Example: (to work out) $\nabla \times \mathbf{r} = 0$

The *Laplacian* $\Delta = \nabla \cdot \nabla$ is a second order derivative. It can operate both on scalar and vector functions. In Cartesian coordinates it is given by

$$\Delta f(\mathbf{r}) = \frac{\partial^2 f(x,y,z)}{\partial^2 x} + \frac{\partial^2 f(x,y,z)}{\partial^2 y} + \frac{\partial^2 f(x,y,z)}{\partial^2 z}$$

3.4 Del operator in cylindrical coordinates

The relation between cylindrical (or polar) and Cartesian coordinates is given by:

$$x = \rho\cos(\varphi),\ y = \rho\sin(\varphi),\ z = z \quad \text{or}\ \rho = \sqrt{x^2+y^2},\ \varphi = \arctan(y/x)$$

This provides

$$\mathrm{d}\mathbf{r} = \underbrace{(\cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y)}_{=\mathbf{e}_\rho(\varphi)}\mathrm{d}\rho + \underbrace{(-\rho\sin(\varphi)\mathbf{e}_x + \rho\cos(\varphi)\mathbf{e}_y)}_{=\rho\mathbf{e}_\varphi(\varphi)}\mathrm{d}\varphi + \mathbf{e}_z\mathrm{d}z$$

The vectors $\mathbf{e}_\rho(\varphi), \mathbf{e}_\varphi(\varphi), \mathbf{e}_z$ are orthonormal. Thus one can write the total differential of Eq. (1) in the form

$$\begin{aligned}\mathrm{d}f =& \frac{\partial f_{cylinder}(\rho,\varphi,z)}{\partial\rho}\mathrm{d}\rho + \frac{\partial f_{cylinder}(\rho,\varphi,z)}{\partial\varphi}\mathrm{d}\varphi + \frac{\partial f_{cylinder}(\rho,\varphi,z)}{\partial z}\mathrm{d}z\\ =& \left(\frac{\partial f_{cylinder}(\rho,\varphi,z)}{\partial\rho}\mathbf{e}_\rho + \frac{\partial f_{cylinder}(\rho,\varphi,z)}{\partial\varphi}\frac{1}{\rho}\mathbf{e}_\varphi + \frac{\partial f_{cylinder}(\rho,\varphi,z)}{\partial z}\mathbf{e}_z\right)\cdot\mathrm{d}\mathbf{r}\end{aligned}$$

and one identifies

$$\nabla = \mathbf{e}_\rho(\varphi)\frac{\partial}{\partial\rho} + \mathbf{e}_\varphi(\varphi)\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \mathbf{e}_z\frac{\partial}{\partial z} \tag{3}$$

In order to obtain the divergence in cylindrical coordinates, we consider a vector function $\mathbf{F}(\mathbf{r}) = \mathbf{e}_\rho(\varphi)F_\rho(\rho,\varphi,z) + \mathbf{e}_\varphi(\varphi)F_\varphi(\rho,\varphi,z) + \mathbf{e}_zF_z(\rho,\varphi,z)$. Now some care is needed, as we need to take into account $\frac{\partial}{\partial\varphi}\mathbf{e}_\rho(\varphi) = \mathbf{e}_\varphi(\varphi)$ and $\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi(\varphi) = -\mathbf{e}_\rho(\varphi)$. Then we find

$$\mathrm{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \underbrace{\frac{\partial F_\rho(\rho,\varphi,z)}{\partial\rho} + \frac{1}{\rho}F_\rho(\rho,\varphi,z)}_{=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho(\rho,\varphi,z))} + \frac{1}{\rho}\frac{\partial F_\varphi(\rho,\varphi,z)}{\partial\varphi} + \frac{\partial F_z(\rho,\varphi,z)}{\partial z}\,. \tag{4}$$

In the same way, we can obtain the curl and the Laplacian. The results are summarized together with the corresponding expressions for spherical coordinates in Tab. 1 on page 11.

3.5 Divergence of $\mathbf{r}/r^3$ and the three-dimensional delta function

The radial-symmetric function $\mathbf{F}(\mathbf{r}) = \mathbf{r}/r^3$ describes, e.g., the electric field of a point charge at the origin. In spherical coordinates we have $\mathbf{F}(\mathbf{r}) = \mathbf{e}_r/r^2$. Using the expression for the divergence in spherical coordinates (see Tab. 1 on page 11), we find $\mathrm{div}\mathbf{F}(\mathbf{r}) = 0/r^2$. Thus, the divergence vanishes for $\mathbf{r} \neq 0$, while the situation at the origin is unclear due to the singularity. In order to resolve this, we consider the function

$$\mathbf{F}_\epsilon(\mathbf{r}) = \frac{r}{r^3+\epsilon^3}\mathbf{e}_r$$

which becomes the original function in the limit $\epsilon \to 0$. we find

$$\mathrm{div}\mathbf{F}_\epsilon(\mathbf{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\frac{r^3}{r^3+\epsilon^3} = \frac{3\epsilon^3}{(r^3+\epsilon^3)^2} = f_\epsilon(\mathbf{r})$$

In the limit $\epsilon \to 0$ the new function $f_\epsilon(\mathbf{r})$ vanishes for $\mathbf{r} \neq 0$, while it becomes infinite at the origin. Integration over the entire three-dimensional space we find

$$\int \mathrm{d}^3r\, f_\epsilon(\mathbf{r}) = 4\pi\int_0^\infty \mathrm{d}r\,\frac{3r^2\epsilon^3}{(r^3+\epsilon^3)^2} = 4\pi\int_0^\infty \mathrm{d}t\,\frac{1}{(t+1)^2} = 4\pi$$

Thus $f_\epsilon(\mathbf{r})/4\pi$ becomes the three-dimensional delta function $\delta(\mathbf{r})$ in the limit $\epsilon \to 0$ and we may write

$$\operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right) = 4\pi\delta(\mathbf{r})\,. \tag{5}$$

Note the difference between the three-dimensional delta function $\delta(\mathbf{r})$ and the one-dimensional delta function $\delta(r)$. While both are infinite at the origin and zero otherwise, we have

$$\int \mathrm{d}^3r\,\delta(\mathbf{r}) = 1 \quad \text{while} \quad \int_{-\infty}^{\infty} \mathrm{d}r\,\delta(r) = 1$$

This immediately shows that the dimension of $\delta(\mathbf{r})$ is inverse volume, while the dimension of $\delta(r)$ is inverse length. Actually, in cartesian coordinates, we have $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$.

3.6 Nabla-calculus

In order to evaluate complicated expression such as $\nabla \cdot f(\mathbf{r})[\mathbf{G}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})]$, one may use the following rules:

1. Mark which functions shall be affected by the differentiation and apply the product rule of differentiation
   $\nabla \cdot f(\mathbf{r})[\mathbf{G}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})] = \nabla \cdot \overset{\downarrow}{f}(\overset{\downarrow}{\mathbf{G}} \times \overset{\downarrow}{\mathbf{H}}) = \nabla \cdot \overset{\downarrow}{f}(\mathbf{G} \times \mathbf{H}) + \nabla \cdot f(\overset{\downarrow}{\mathbf{G}} \times \mathbf{H}) + \nabla \cdot f(\mathbf{G} \times \overset{\downarrow}{\mathbf{H}})$
2. Transform the expression according to the rules of vector algebra where ∇ is treated as a conventional vector (keep the arrows!).
   $= \overset{\downarrow}{f}\nabla \cdot (\mathbf{G} \times \mathbf{H}) + f\mathbf{H} \cdot (\nabla \times \overset{\downarrow}{\mathbf{G}}) + f\mathbf{G} \cdot (\overset{\downarrow}{\mathbf{H}} \times \nabla)$
   Here the triple-scalar-product rule was used for the second and third term.
3. Order the expressions such, that only the marked functions are on the right-hand side of ∇. Then the markings can be finally abandoned.
   $= (\mathbf{G} \times \mathbf{H}) \cdot \nabla f + f\mathbf{H} \cdot (\nabla \times \mathbf{G}) - f\mathbf{G} \cdot (\nabla \times \mathbf{H})$

Examples: (to work out)

$$\operatorname{div} \frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} = 0 \quad \text{for} \quad \mathbf{r} \neq \mathbf{a} \tag{6}$$

$$\nabla \times \frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} = 0 \quad \text{for} \quad \mathbf{r} \neq \mathbf{a} \tag{7}$$

4 Integrals in three dimensions

4.1 Line integrals

Example: For a given path of a particle the work acted on the particle reads

$$W = \sum_{\text{small path elements}\Delta r_i} \Delta\mathbf{r}_i \cdot \mathbf{F}(\mathbf{r}_i) \xrightarrow{\text{elements}\to 0} \int_{\text{path}} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r})$$

In order to evaluate the line integral we parameterize the path $\mathbf{r}(s)$ with $s_1 < s < s_2$. Then $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\mathrm{d}s$ and

$$W = \int_{s_1}^{s_2} \mathrm{d}s\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \cdot \mathbf{F}(\mathbf{r}(s))$$

A special parameterization is the time t. Then $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$ is the velocity and $\mathbf{v} \cdot \mathbf{F}(\mathbf{r}(t))$ the power, such that the total applied work is the time integral of mechanical power.

In particular we write for a closed path which is the boundary of an area S

$$W = \oint_{\partial\mathcal{S}} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r})$$

4.2 Volume integral

Example: Consider molecules within a volume $\mathcal{V}$ of a liquid.

$$N = \sum_{\text{all small partial volumes at}\mathbf{r}_i} n(\mathbf{r}_i)\Delta V_i \xrightarrow{\text{partial volumes}\to 0} \int_{\mathcal{V}} \mathrm{d}V\, n(\mathbf{r}) \quad \left(\text{also} \int_{\mathcal{V}} \mathrm{d}^3 r\, n(\mathbf{r})\right)$$

Here we need to parameterize the space by three parameters s, t, u such that for $(s, t, u) \in$ `Vrange`, the points $\mathbf{r}(s, t, u)$ are within $\mathcal{V}$. The the volume element is $\mathrm{d}V = \left|\frac{\partial \mathbf{r}}{\partial s} \cdot \left(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial u}\right)\right| \mathrm{d}s\mathrm{d}t\mathrm{d}u$ and we find

$$N = \int_{\texttt{Vrange}} \mathrm{d}s\mathrm{d}t\mathrm{d}u \left|\frac{\partial \mathbf{r}}{\partial s} \cdot \left(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial u}\right)\right| n(\mathbf{r}(s, t, u))$$

To work out: $\int_{\text{cylinder}} \mathrm{d}^3 r\, z$ in Cartesian and cylinder coordinates

4.3 Surface integral

Example: Particle flow through a surface $\mathcal{S}$. The surface is decomposed in small surface elements $\Delta\mathbf{S}_i$, where the vector is pointing perpendicular to the surface and defines the orientation of the surface.

$$I = \sum_{\text{all surface elements}} \Delta\mathbf{S}_i \cdot \mathbf{j}(\mathbf{r}_i) \xrightarrow{\text{surface elements}\to 0} \int_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \mathbf{j}(\mathbf{r})$$

The surface is parameterized by two parameters u, v. If $(u, v) \in$ `Srange`, the points $\mathbf{r}(u, v)$ are within $\mathcal{S}$. Then $\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ and we obtain

$$I = \int_{\texttt{Srange}} \mathrm{d}u\mathrm{d}v \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \cdot \mathbf{j}(\mathbf{r}(u, v))$$

In particular the surface enclosing the volume $\mathcal{V}$ is written as $\mathcal{S} = \partial\mathcal{V}$ and the integral is written as

$$I = \oint_{\partial\mathcal{V}} \mathrm{d}\mathbf{S} \cdot \mathbf{j}(\mathbf{r})$$

4.4 Gauss' Theorem

$$\oint_{\partial\mathcal{V}} \mathrm{d}\mathbf{S} \cdot \mathbf{F}(\mathbf{r}) = \int_{\mathcal{V}} \mathrm{d}V\ \mathrm{div}\,\mathbf{F}(\mathbf{r})$$

Proof for a cuboid with edges L_x, L_y, L_z

$$\begin{aligned}
\int_{\mathcal{V}} \mathrm{d}V \ \mathrm{div}\, \mathbf{F}(\mathbf{r}) &= \int_0^{L_x} \mathrm{d}x \int_0^{L_y} \mathrm{d}y \int_0^{L_z} \mathrm{d}z \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \\
&= \int_0^{L_y} \mathrm{d}y \int_0^{L_z} \mathrm{d}z [F_x(L_x, y, z) - F_x(0, y, z)] + 2 \text{ further terms} \\
&= \int_{\text{face at } x=L_x} \mathrm{d}\mathbf{S} \cdot \mathbf{F}(\mathbf{r}) + \int_{\text{face at } x=0} \mathrm{d}\mathbf{S} \cdot \mathbf{F}(\mathbf{r}) + 4 \text{ further surfaces} \\
&= \int_{\text{face of cuboid}} \mathrm{d}\mathbf{S} \cdot \mathbf{F}(\mathbf{r})
\end{aligned}$$

Here the orientation of all faces is directed outwards of the cuboid, which compensates the minus sign at $x = 0$. □

4.5 Continuity equation

Consider particles within a volume $\mathcal{V}$ of a fluid. The number of particles N is diminished by a current density through the surface of the volume and increased by sources $q(\mathbf{r})$ (particle production per volume at location $\mathbf{r}$.) Thus:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\oint_{\partial\mathcal{V}} \mathrm{d}\mathbf{S} \cdot \mathbf{j}(\mathbf{r}, t) + \int_{\mathcal{V}} \mathrm{d}V\, q(\mathbf{r}, t)$$

With $N(t) = \int_{\mathcal{V}} \mathrm{d}V\, n(\mathbf{r}, t)$ and Gauss' theorem we find

$$\int_{\mathcal{V}} \mathrm{d}V \left[\frac{\partial n(\mathbf{r}, t)}{\partial t} + \mathrm{div}\, \mathbf{j}(\mathbf{r}, t) \right] = \int_{\mathcal{V}} \mathrm{d}V\, q(\mathbf{r}, t)$$

As this relation holds for arbitrary volumes, this provides the

continuity equation $\quad \dfrac{\partial n(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r}, t) = q(\mathbf{r}, t)$

For liquids the density n is constant and we find $\nabla \cdot \mathbf{j}(\mathbf{r}, t) = n\nabla \cdot \mathbf{v}(\mathbf{r}, t) = 0$. Thus fields with vanishing divergence are often called incompressible. Another common denotation is *solenoidal* (as the magnetic field satisfies $\nabla \cdot \mathbf{B} = 0$).

4.6 Stokes' Theorem

$$\oint_{\partial\mathcal{S}} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) = \int_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \nabla\times \mathbf{F}(\mathbf{r})$$

Proof for a rectangle in the x, y-plane (i.e. $z = 0$) with edges L_x, L_y

$$\begin{aligned}
\int_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \nabla\times \mathbf{F}(\mathbf{r}) &= \int_0^{L_x} \mathrm{d}x \int_0^{L_y} \mathrm{d}y\, \mathbf{e}_z \cdot \nabla\times \mathbf{F}(\mathbf{r}) = \int_0^{L_x} \mathrm{d}x \int_0^{L_y} \mathrm{d}y \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\
&= \int_0^{L_y} \mathrm{d}y\, [F_y(L_x, y, 0) - F_y(0, y, 0)] - \int_0^{L_x} \mathrm{d}x\, [F_x(x, L_y, 0) - F_x(x, 0, 0)] \\
&= \int_0^{L_x} \mathrm{d}x\, F_x(x, 0, 0) + \int_0^{L_y} \mathrm{d}y\, F_y(L_x, y, 0) + \int_{L_x}^0 \mathrm{d}x\, F_x(x, L_y, 0) + \int_{L_y}^0 \mathrm{d}y\, F_y(0, y, 0) \\
&= \oint_{\partial\mathcal{S}} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) \quad \square
\end{aligned}$$

In hydrodynamics a finite integral $\oint_{\partial\mathcal{S}} \mathrm{d}\mathbf{r}\cdot\mathbf{v}(\mathbf{r}) \neq 0$ denotes a rotation in a fluid. Thus the closed line integral is called circulation and $\nabla \times \mathbf{v}(\mathbf{r})$ is denoted as circulation density or vorticity. Fields with $\nabla \times \mathbf{F} = 0$ are called *irrotational*.

5 Potentials

5.1 Scalar Potential

Let $\mathbf{F}(\mathbf{r})$ be an arbitrary vector function, which is defined in the simply connected region (i.e. there are no holes) of space $\mathcal{V}$. Then

$$\underbrace{f(\mathbf{r}) \quad \text{exists with} \quad \mathbf{F}(\mathbf{r}) = -\nabla f(\mathbf{r}) \quad \text{in}\,\mathcal{V}}_{\text{a scalar potential exists}} \quad \Leftrightarrow \quad \underbrace{\nabla \times \mathbf{F}(\mathbf{r}) = 0 \quad \text{in} \quad \mathcal{V}}_{\mathbf{F}(\mathbf{r}) \text{ is irrotational}}$$

The scalar potential $f(\mathbf{r})$ is uniquely determined by $\mathbf{F}(\mathbf{r})$ except for an arbitrary constant.

Proof:

$\Rightarrow$ $\nabla\times(\nabla f(\mathbf{r})) = \nabla\times(\nabla \overset{\downarrow}{f}) = (\nabla\times\nabla)\overset{\downarrow}{f} = 0$. Alternatively one may use Cartesian coordinates and apply the symmetry of second derivatives.

$\Leftarrow$ For a given $\mathbf{r}_0 \in \mathcal{V}$ we define

$$f(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathrm{d}\mathbf{r}\cdot\mathbf{F}(\mathbf{r})$$

The integral does not depend on the particular path. Otherwise one can construct a closed line integral from the two paths with non-vanishing circulation, which is excluded by Stokes' theorem and $\nabla \times \mathbf{F}(\mathbf{r}) = 0$. Now

$$\mathrm{d}f(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathrm{d}\mathbf{r}\cdot\mathbf{F}(\mathbf{r}) + \int_{\mathbf{r}_0}^{\mathbf{r}} \mathrm{d}\mathbf{r}\cdot\mathbf{F}(\mathbf{r}) = -\mathrm{d}\mathbf{r}\cdot\mathbf{F}(\mathbf{r})$$

and we identify $\nabla f(\mathbf{r}) = -\mathbf{F}(\mathbf{r})$ from Eq. (1). Thus $f(\mathbf{r})$ is the scalar potential. □

5.2 Vector Potential

Let $\mathbf{F}(\mathbf{r})$ be an arbitrary vector function, which is defined in the simply connected region of space $\mathcal{V}$. Then

$$\underbrace{\mathbf{A}(\mathbf{r}) \quad \text{exists with} \quad \mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) \quad \text{in}\,\mathcal{V}}_{\text{a vector potential exists}} \quad \Leftrightarrow \quad \underbrace{\operatorname{div}\mathbf{B}(\mathbf{r}) = 0 \quad \text{in} \quad \mathcal{V}}_{\mathbf{B}(\mathbf{r}) \quad \text{is incompressible/solenoidal}}$$

There is some freedom to choose the vector potential $\mathbf{A}(\mathbf{r})$. Two potential $\mathbf{A}_1(\mathbf{r})$ and $\mathbf{A}_2(\mathbf{r})$ provide the identical function $\mathbf{F}(\mathbf{r})$ if $\mathbf{A}_1(\mathbf{r}) - \mathbf{A}_2(\mathbf{r}) = \operatorname{grad}\xi(\mathbf{r})$. (gauge invariance)

Proof:

$\Rightarrow$ $\nabla(\nabla\times\mathbf{A}(\mathbf{r})) = \nabla\cdot(\nabla\times\overset{\downarrow}{\mathbf{A}}) = \overset{\downarrow}{\mathbf{A}}\cdot(\nabla\times\nabla) = (\nabla\times\nabla)\cdot\mathbf{A} = 0.$

$\Leftarrow$ We construct $\mathbf{A}(\mathbf{r})$ in Cartesian coordinates. Due to the gauge invariance we may set $A_z = 0$. Then

$$B_x(x,y,z) = -\frac{\partial A_y}{\partial z} \Rightarrow A_y(x,y,z) = -\int_{z_0}^{z} \mathrm{d}z'\, B_x(x,y,z') + f(x,y)$$

$$B_y(x,y,z) = \frac{\partial A_x}{\partial z} \Rightarrow A_x(x,y,z) = \int_{z_0}^{z} \mathrm{d}z'\, B_y(x,y,z')$$

$$B_z(x,y,z) = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\int_{z_0}^{z} \mathrm{d}z' \underbrace{\left(\frac{\partial B_x(x,y,z')}{\partial x} + \frac{\partial B_y(x,y,z')}{\partial y}\right)}_{=-\frac{\partial B_z(x,y,z')}{\partial z'}} + \frac{\partial f(x,y)}{\partial x}$$

$$= B_z(x,y,z) - B_z(x,y,z_0) + \frac{\partial f(x,y)}{\partial x}$$

By choosing $f(x,y) = \int_{x_0}^{x} \mathrm{d}x'\, B_z(x',y,z_0)$ the last equation is satisfied. Thus we have constructed the vector potential

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \int_{z_0}^{z} \mathrm{d}z'\, B_y(x,y,z') \\ -\int_{z_0}^{z} \mathrm{d}z'\, B_x(x,y,z') + \int_{x_0}^{x} \mathrm{d}x' B_z(x',y,z_0) \\ 0 \end{pmatrix}$$

5.3 Defining a vector field by divergence and curl

For given functions $q(\mathbf{r}), \mathbf{w}(\mathbf{r})$, and $h(\mathbf{r})$, the vector field $\mathbf{F}(\mathbf{r})$ is uniquely defined for $\mathbf{r} \in \mathcal{V}$ (simply connected) by the requirements

$$\nabla \cdot \mathbf{F} = q(\mathbf{r})\,, \quad \nabla \times \mathbf{F} = \mathbf{w}(\mathbf{r}) \quad \text{and} \quad \mathbf{n} \cdot \mathbf{F} = h(\mathbf{r}) \quad \text{on} \quad \partial\mathcal{V}$$

The construction contains three parts which add up to the vector field $\mathbf{F}(\mathbf{r})$:

$$\begin{aligned} \mathbf{F}_l &= -\nabla \int \mathrm{d}^3 r' \, \frac{q(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \\ \mathbf{F}_t &= \nabla \times \int \mathrm{d}^3 r' \, \frac{\mathbf{w}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \\ \mathbf{F}^{\text{bound}} &= \nabla\Psi \quad \text{with} \quad \Delta\Psi = 0 \quad \text{and} \quad \mathbf{n}\cdot\nabla\Psi = h(\mathbf{r}) - \mathbf{n}\cdot(\mathbf{F}_l + \mathbf{F}_t) \quad \text{on} \quad \partial\mathcal{V} \end{aligned}$$

In Fourier space we find $\mathbf{F}_l(\mathbf{q}) \parallel \mathbf{q}$ and $\mathbf{F}_t(\mathbf{q}) \perp \mathbf{q}$. The term $\mathbf{F}^{\text{bound}}$ is both irrotational and solenoidal. If $\mathcal{V}$ is the entire three-dimensional space and we apply the boundary condition $\mathbf{F} \to 0$ for $r \to \infty$, we find $\mathbf{F}^{\text{bound}} = 0$, provided that $g(\mathbf{r})$ and $\mathbf{w}(\mathbf{r})$ are restricted to a finite region in space.

	Cartesian coordinates	polar coordinates	spherical coordinates
Coordinates q_i	(x, y, z)	(ρ, φ, z)	(r, θ, φ)
$x =$	x	$\rho\cos(\varphi)$	$r\sin(\theta)\cos(\varphi)$
$y =$	y	$\rho\sin(\varphi)$	$r\sin(\theta)\sin(\varphi)$
$z =$	z	z	$r\cos(\theta)$
Unit vector $\mathbf{e}_{q_1} =$	$\mathbf{e}_x$	$\mathbf{e}_\rho = \cos\varphi\mathbf{e}_x + \sin\varphi\mathbf{e}_y$	$\mathbf{e}_r = \sin\theta\cos\varphi\mathbf{e}_x + \sin\theta\sin\varphi\mathbf{e}_y + \cos\theta\mathbf{e}_z$
Unit vector $\mathbf{e}_{q_2} =$	$\mathbf{e}_y$	$\mathbf{e}_\varphi = -\sin\varphi\mathbf{e}_x + \cos\varphi\mathbf{e}_y$	$\mathbf{e}_\theta = \cos\theta\cos\varphi\mathbf{e}_x + \cos\theta\sin\varphi\mathbf{e}_y - \sin\theta\mathbf{e}_z$
Unit vector $\mathbf{e}_{q_3} =$	$\mathbf{e}_z$	$\mathbf{e}_z$	$\mathbf{e}_\varphi = -\sin\varphi\mathbf{e}_x + \cos\varphi\mathbf{e}_y$
del operator: $\nabla =$	$\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}$	$\mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \mathbf{e}_z\frac{\partial}{\partial z}$	$\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$
Gradient: $\nabla f(\mathbf{r}) =$	$\mathbf{e}_x\frac{\partial f}{\partial x} + \mathbf{e}_y\frac{\partial f}{\partial y} + \mathbf{e}_z\frac{\partial f}{\partial z}$	$\mathbf{e}_\rho\frac{\partial f}{\partial\rho} + \mathbf{e}_\varphi\frac{1}{\rho}\frac{\partial f}{\partial\varphi} + \mathbf{e}_z\frac{\partial f}{\partial z}$	$\mathbf{e}_r\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}$
Divergence: $\nabla\cdot\mathbf{F}(\mathbf{r}) =$	$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$
Curl: $\nabla\times\mathbf{F}(\mathbf{r}) =$	$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{e}_x$ $+\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{e}_y$ $+\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{e}_z$	$\left(\frac{1}{\rho}\frac{\partial F_z}{\partial\varphi} - \frac{\partial F_\varphi}{\partial z}\right)\mathbf{e}_\rho$ $+\left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\mathbf{e}_\varphi$ $+\frac{1}{\rho}\left(\frac{\partial(\rho F_\varphi)}{\partial\rho} - \frac{\partial F_\rho}{\partial\varphi}\right)\mathbf{e}_z$	$\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta F_\varphi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\varphi}\right)\mathbf{e}_r$ $+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial(rF_\varphi)}{\partial r}\right)\mathbf{e}_\theta$ $+\frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\mathbf{e}_\varphi$
Laplacian: $\Delta f =$	$\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f_\rho}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial^2\varphi} + \frac{\partial^2 f}{\partial^2 z}$	$\frac{1}{r}\frac{\partial^2(rf)}{\partial^2 r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial^2\theta} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial^2\varphi}$
Volume element: d^3r	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$	$\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z$	$\sin\theta r^2\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$

Table 1: Vector analysis in different coordinate systems. The unit vectors are defined as $\mathbf{e}_q = \frac{\partial\mathbf{r}}{\partial q}/|\frac{\partial\mathbf{r}}{\partial q}|$. They are ordered such such that $\mathbf{e}_{q_1}, \mathbf{e}_{q_2}, \mathbf{e}_{q_3}$ form an orthonormal basis with positive orientation in each case.