Quadratic air drag (MÖ 21 September 2005)

To check that your programs work correct, you can plot equation (9) below in the same figure as your numerical result for the special case of $v_0 = 0$ *and* $\theta = -\pi/2$.

When the air drag is a quadratic function of the velocity

$$F = mg - cv\,(t)^2 \geq 0 \tag{1}$$

we can use Newton's second law

$$F = ma = m\frac{dv\,(t)}{dt}$$

to get the following differential equation

$$\frac{dv\,(t)}{dt} = g - \frac{c}{m}v\,(t)^2\,. \tag{2}$$

After rewriting (2) in the following way

$$\frac{\frac{dv(t)}{dt}}{g - \frac{c}{m}v\,(t)^2} = \frac{\frac{dv(t)}{dt}}{g\left(1 - k^2 v\,(t)^2\right)} = 1,\; k^2 = \frac{c}{mg}$$

we can divide it into two parts according to

$$\frac{2\frac{dv(t)}{dt}}{\left(1 - k^2 v\,(t)^2\right)} = \frac{2\frac{dv(t)}{dt}}{(1 + kv\,(t))\,(1 - kv\,(t))} = \frac{\frac{dv(t)}{dt}}{(1 + kv\,(t))} + \frac{\frac{dv(t)}{dt}}{(1 - kv\,(t))} = 2g. \tag{3}$$

We now integrate (3) with respect to time

$$\frac{1}{k}\ln|1 + kv\,(t)| - \frac{1}{k}\ln|1 - kv\,(t)| = 2gt + C.$$

The absolute sign in the argument of the first $\ln -$term can be taken away since $k,\ v\,(t) \geq 0$. And from (1) follows that $(1 + kv\,(t))\,(1 - kv\,(t)) \geq 0$, so we can skip also the absolute sign in the argument of the second $\ln -$term. With help of the logarithmic laws, we can then come to

$$\ln\left(\frac{1 + kv\,(t)}{1 - kv\,(t)}\right) = 2kgt + kC,$$

so that

$$\exp\left(\ln\left(\frac{1 + kv\,(t)}{1 - kv\,(t)}\right)\right) = \frac{1 + kv\,(t)}{1 - kv\,(t)} = \exp\left(2kgt + kC\right) \equiv f\,(t)\,. \tag{4}$$

Some simplifications give

$$1 + kv(t) = (1 - kv(t)) f(t) \ \Rightarrow \ v(t) k (1 + f(t)) = f(t) - 1 \tag{5}$$

By applying the initial condition $v(0) = 0$ we get $f(0) = 1$, we then see from (4) that $kC = 0$ such that

$$f(t) = \exp\left(2\sqrt{\frac{cg}{m}}t\right). \tag{6}$$

From (5), applying (6) we when have a closed expression for the velocity

$$v(t) = \frac{1}{k}\frac{f(t) - 1}{f(t) + 1} = \sqrt{\frac{mg}{c}}\frac{\exp\left(2\sqrt{\frac{cg}{m}}t\right) - 1}{\exp\left(2\sqrt{\frac{cg}{m}}t\right) + 1} = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}t\right). \tag{7}$$

From the properties of the $\tanh(x)$ function

$$\begin{cases} \lim_{x\to\infty}\tanh(x) = 1 \\ \lim_{x\to 0}\tanh(x) \sim x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + ... \end{cases}$$

we can also obtain information about some limit cases.

If we take the limit of $t \to \infty$, (7) is approaching the value $\sqrt{mg/c}$, which is the highest velocity a free falling object with quadratic air drag of strength c will obtain. If we take the limit of $c \to 0$, we get $v(t) \to gt$, the velocity of a free falling object. It is now straightforward to integrate (7) with respect to time, in order to obtain $s(t)$

$$s(t) = \int v(t)\,dt = \frac{m}{c}\ln\left(\cosh\left(\sqrt{\frac{cg}{m}}t\right)\right). \tag{8}$$

This primitive function directly fulfill the initial condition $s(0) = 0$. The free fall result can again be checked in the limit of small c. With help of the following taylor expansions

$$\begin{cases} \lim_{x\to 0}\cosh(x) \sim 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + ... \\ \lim_{x\to 0}\ln(\cosh(x)) \sim \frac{1}{2}x^2 - \frac{1}{12}x^4 + ... \end{cases},$$

one receive from (8), $s(t) \to gt^2/2$ in the limit of $c \to 0$.

Summery

The problem of a falling object with the air drag being quadratically dependent of the velocity (with strength $c \geq 0$) have an analytic solution

$$\begin{cases} v(t) = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}t\right) \\ s(t) = \frac{m}{c}\ln\left(\cosh\left(\sqrt{\frac{cg}{m}}t\right)\right) \end{cases}. \tag{9}$$