TASK #1:

Perturbation theory: Energy conservation: $\mathcal{E}_{\mathbb{k}} - \mathcal{E}_g = \Omega \pm \omega$

$\begin{pmatrix} \text{Neglect prefactors} \\ A_0 \equiv \Lambda \end{pmatrix}$

$$M^{(1)}_{\mathbb{k}} = \int d\mathbb{k}' \frac{\langle \mathbb{k} | \hat{k}_z | \mathbb{k}' \rangle \langle \mathbb{k}' | \hat{k}_z | g \rangle}{\mathcal{E}_g + \Omega - \mathcal{E}_{\mathbb{k}'} + i\eta} =$$

$$= \int d\mathbb{k}' \, \underbrace{k_z}_{k\cos\theta_k} \frac{\overbrace{\langle \mathbb{k} | \mathbb{k}' \rangle}^{\delta(\mathbb{k}-\mathbb{k}')} \langle \mathbb{k}' | \hat{k}_z | g \rangle}{\mathcal{E}_g + \Omega - \mathcal{E}_{\mathbb{k}'}}$$

Should not change sign with akslewis acc. Rattenbacher ... Wörner (2018)

$$= k_z \frac{\langle \mathbb{k} | \hat{k}_z | g \rangle}{\mathcal{E}_g + \Omega - \mathcal{E}_{\mathbb{k}}} = \frac{k_z \langle \mathbb{k} | \hat{k}_z | g \rangle}{\mathcal{E}_g + \Omega - \Omega \mp \omega - \mathcal{E}_g} = \mp \frac{k_z}{\omega} \langle \mathbb{k} | \hat{k}_z | g \rangle$$

Approximate dipole operator to continuum from ground state:

$$\langle \mathbb{k} | \hat{k}_z | g \rangle = k_z \langle \mathbb{k} | g \rangle$$

$$M^{(2)}_{\mathbb{k}} \approx \frac{k_z^2}{\omega} \langle \mathbb{k} | g \rangle = \frac{k^2 \cos^2\theta_k}{\omega} \langle \mathbb{k} | g \rangle = 2 \frac{\mathcal{E}}{\omega} \cos^2\theta_k \langle \mathbb{k} | g \rangle$$

Given $\phi_{\mathbb{k}} = \frac{1}{(2\pi)^{3/2}} e^{i\mathbb{k}\cdot\mathbb{r}}$ $\qquad g = \frac{Z^{3/2}}{\pi^{1/2}} e^{-Zr}$

TASK #2

$$\langle \mathbb{k} | g \rangle = \frac{1}{(2\pi)^{3/2}} \frac{Z^{3/2}}{\pi^{1/2}} \int d^3r \, e^{-i\mathbb{k}\cdot\mathbb{r}} e^{-Zr}$$

angle between $\mathbb{r}$ and $\mathbb{k}$

$$\int d^3r \, e^{-ikr\cos\theta} e^{-Zr} = \int_0^\infty dr\, r^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \, e^{-ikr\cos\theta} e^{-Zr}$$

$$= \int_0^\infty dr\, r^2 2\pi \frac{1}{+ikr} \underbrace{\left[e^{-ikr\cos\theta} \right]\Big|_0^\pi}_{e^{+ikr} - e^{-ikr}} e^{-Zr} = \frac{2\pi}{ik} \int_0^\infty dr\, r \left(e^{-(Z-ik)r} - e^{-(Z+ik)r} \right)$$

Integration by parts:

$$\int_0^\infty dr\, r e^{-\alpha r} = \underbrace{\left[r\frac{e^{-\alpha r}}{-\alpha} \right]\Big|_0^\infty}_{0-0} - \int_0^\infty dr \frac{e^{-\alpha r}}{-\alpha} = -\frac{1}{\alpha^2} \underbrace{\left[e^{-\alpha r} \right]\Big|_0^\infty}_{0-1} = \frac{1}{\alpha^2}$$

$$= \frac{2\pi}{ik} \left\{ \frac{1}{(Z-ik)^2} - \frac{1}{(Z+ik)^2} \right\} = \frac{2\pi}{ik} \left\{ \frac{(Z+ik)^2 - (Z-ik)^2}{(Z^2+k^2)^2} \right\} = \frac{2\pi \cdot 4ikZ}{ik(Z^2+k^2)^2}$$

$$= \frac{8\pi Z}{(Z^2+k^2)^2}$$

$$\langle \mathbb{k}|g\rangle = \frac{1}{(2\pi)^{3/2}} \cdot \frac{Z^{3/2}}{\pi^{1/2}} \cdot \frac{8\pi Z}{(Z^2+k^2)^2} = \frac{2^{3/2}}{\pi} \cdot \frac{Z^{5/2}}{(Z^2+k^2)^2}$$

$$\begin{cases} I_p = Z^2/2 \iff Z = I_p^{1/2}\, 2^{1/2} \rightarrow Z^{5/2} = I_p^{5/4}\, 2^{5/4} \\ \mathcal{E} = k^2/2 \iff k^2 = 2\mathcal{E} \end{cases}$$

$$\langle \mathbb{k}|g\rangle = \frac{2^{\overbrace{\frac{6}{4}+\frac{5}{4}-\frac{8}{4}}^{3/4}}}{\pi} \frac{I_p^{5/4}}{(I_p+\mathcal{E})} = \frac{2^{3/4}}{\pi} \frac{I_p^{5/4}}{(I_p+\mathcal{E})^2} \quad \blacksquare$$

TASK#3

Ansatz: $\psi_{\mathbb{k}}^{V} = \phi_{\mathbb{k}}(\mathbb{r})\, e^{-i\Phi_{\mathbb{k}}(t)}$ into TDSE: $i\frac{\partial}{\partial t}\psi_{\mathbb{k}}^{V} = \frac{1}{2}[\mathbb{p}+\mathbb{A}(t)]^2 \psi_{\mathbb{k}}^{V}$

LHS: $i\frac{\partial}{\partial t}\phi_{\mathbb{k}}(\mathbb{r})\, e^{-i\Phi_{\mathbb{k}}(t)} = +\phi_{\mathbb{k}}(\mathbb{r}) \frac{\partial \Phi_{\mathbb{k}}}{\partial t} e^{-i\Phi_{\mathbb{k}}(t)}$

RHS: $\underbrace{\frac{1}{2}[\mathbb{p}+\mathbb{A}(t)]^2 \phi_{\mathbb{k}}(\mathbb{r})}_{\hat{\mathbb{p}}\phi_{\mathbb{k}} = \mathbb{k}\phi_{\mathbb{k}}} e^{-i\Phi(t)} = \frac{1}{2}[\mathbb{k}+\mathbb{A}(t)]^2 \phi_{\mathbb{k}}(\mathbb{r})\, e^{-i\Phi(t)}$

Cancel $\phi_{\mathbb{k}}(\mathbb{r})\, e^{-i\Phi_{\mathbb{k}}(t)}$

$$\Rightarrow \frac{\partial \Phi_{\mathbb{k}}}{\partial t} = \frac{1}{2}[\mathbb{k}+\mathbb{A}(t)]^2 \xrightarrow{\text{Integrate}} \Phi_{\mathbb{k}}(t) = \int_{\text{ref.}}^{t} dt' \frac{1}{2}[\mathbb{k}+\mathbb{A}(t)]^2 \quad \blacksquare$$

TASK#4 Show photon picture emerges from timedependent interaction

$$C_{\mathbb{k}}(t) \simeq \frac{1}{i}\langle \varphi_{\mathbb{k}}|\hat{p}_z|g\rangle \int_{-\infty}^{t} dt' A_x(t') \exp\left[i\int^{t'} dt'' \left\{\frac{\mathbb{k}^2}{2} + I_p + \mathbb{k}\cdot\mathbb{A}_L \sin\omega_L t''\right\}\right]$$

$$\exp\left[i\left(\frac{\mathbb{k}^2}{2}+I_p\right)t'\right] \exp\left[-i\frac{\mathbb{k}\cdot\mathbb{A}_L}{\omega_L}\cos\omega_L t'\right] = \left\{\text{Bessel function: } e^{iz\cos\theta} = \sum_{n=-\infty}^{+\infty} i^n J_n(z) e^{in\theta}\right\}$$

$$\left\{\text{Negative argument: } J_n(-z) = J_n(z)\, \overbrace{e^{i\pi n}}^{(-1)^n}\right\} = \exp\left[i\left(\frac{\mathbb{k}^2}{2}+I_p\right)t'\right] \sum_{n=-\infty}^{+\infty} \underbrace{(-1)^n i^n}_{(-i)^n} J_n\left(\frac{\mathbb{k}\cdot\mathbb{A}_L}{\omega_L}\right) e^{in\omega_L t'}$$

$$= \sum_{n=-\infty}^{+\infty} (-i)^n \exp\left[i\left(\frac{\mathbb{k}^2}{2}+I_p+n\omega_L\right)t'\right] J_n\left(\frac{\mathbb{k}\cdot\mathbb{A}_L}{\omega_L}\right)$$

$$C_{\mathbf{k}}(t) \approx \frac{1}{i}\langle e_{\mathbf{k}}|\hat{p}_z|g\rangle \int_{-\infty}^{t} dt' \frac{\Lambda_x(t')}{2} \sum_{n=-\infty}^{+\infty} (-i)^n J_n\left(\frac{\mathbf{k}\cdot\Lambda_L(t')}{\omega_L}\right) \ldots$$

$$\ldots \times \exp[i(\varepsilon_{\mathbf{k}} + I_p + n\omega_L - \omega_x)t']$$

$$A_x(t) \approx \frac{1}{2}\Lambda_x(t) e^{-i\omega_x t}$$

(neglect emission term)

(When n is positive ε_k can be smaller and still be resonant with $I_p - \omega_x$. So $n > 0$ is emission of n-photons)

(Low intensity approximation: $J_n(z) \approx \left(\frac{1}{2}z\right)^n / \underbrace{\Gamma(n+1)}_{n!}$

$$J_1\left(\frac{k\Lambda_L}{\omega_L}\right) \approx \left(\frac{1}{2}\frac{k\Lambda_L}{\omega_L}\right)^1 / 1! = \frac{1}{2}\frac{\mathbf{k}\Lambda_L}{\omega_L} = \frac{1}{2}\Lambda_L \frac{k_z}{\omega_L}$$
)

(So we recover that the continuum-continuum transition grows with k_z and with smaller laser frequency $1/\omega_L$)

(The sign of $n = \pm 1$ are different due to the $(-i)^n$ factor. but there is also a $J_{-n}(z) = (-1)^n J_n(z)$ factor so the $(-1)^n$ will be cancelled. Left is i^n $\begin{cases} i^1 \\ i^{-1} = -i \end{cases}$ differ by π.
(anyway $(-1)^n = (-1)^{-n}$)
)

TASK #5 Assume short pulse $A_x = \Lambda_x(t-t_0)\sin\omega_x t \approx \frac{\Lambda_x}{2}(t-t_0)\exp[-i\omega_x t]$

$$C_{\mathbf{k}}(t) \approx \frac{1}{i}\langle e_{\mathbf{k}}|\hat{p}_z|g\rangle \int_{-\infty}^{t} dt' \frac{1}{2}\Lambda_x(t'-t_0) \cdot \exp\left[i\int^{t'} dt''\left(\frac{(\mathbf{k}+A_L(t''))^2}{2} + I_p - \omega_x\right)\right].$$

$$\exp\left[i\left(\int^{t'} dt'' \frac{(\mathbf{k}+A_L(t''))^2}{2} + I_p - \omega_x\right)\right]$$

$$\exp\left[i\int_{t_0}^{t'} dt''\left(\frac{(\mathbf{k}+A_L(t''))^2}{2} + I_p - \omega_x\right) + \int^{t_0} dt'' \ldots\right]$$

$$\approx \exp\left[i\int_{t_0}^{t'} dt''\left(\frac{(\mathbf{k}+A_L(t_0))^2}{2} + I_p - \omega_x\right) + \int^{t_0} \ldots\right]$$

$$\exp\left[i\left(\frac{(\mathbf{k}+A_L)^2}{2} + I_p - \omega_x\right)(t'-t_0) + \int^{t_0} \ldots\right]$$

$$\approx \exp\left[i\left(\frac{k^2}{2} + I_p - \omega_x + \mathbf{k}\cdot A_L(t_0)\right)t' + \text{const. phase}\right]$$

TASK#6:

$$f(E) = \int_{-\infty}^{+\infty} dt \; F(t)\, e^{+iEt} = \int_{-\infty}^{+\infty} dt \left\{ i\delta(t) + \frac{\Gamma}{2}(q-i)\, e^{-iE_r t - \Gamma t/2}\, \theta(t) \right\} e^{iEt}$$

$$= i + \left[\frac{\frac{\Gamma}{2}(q-i)}{(-iE_r - \Gamma/2 + iE)} e^{(-iE_r - \Gamma/2 + iE)t} \right]_0^{\infty}$$

$$= i + i\,\frac{\frac{\Gamma}{2}(q-i)}{(-E+E_r - i\Gamma/2)}(0-1) = \left(1 + \frac{(q-i)}{(+\varepsilon + i)}\right) i$$

$$= \left(\frac{\varepsilon + \cancel{i} + q - \cancel{i}}{\varepsilon + i} \right) i = \left(\frac{\varepsilon+q}{\varepsilon+i}\right) i = \left(\frac{\varepsilon + q}{1 - i\varepsilon}\right)\cancel{\frac{i}{i}}$$

Task#7: $q=0$: Lorentzian

$$\arg\left(\frac{\varepsilon}{1-i\varepsilon}\right) = \arg\left(\frac{\varepsilon}{1+\varepsilon^2}(1+i\varepsilon)\right) \approx \varepsilon = \frac{E-E_r}{\Gamma/2}$$